THE FIRST TRANSLATION OF EUCLID'S ELEMENTS INTO ENGLISH AND ITS SOURCE

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1. Introductory. Note 74 of my recently published *Outline of the History of Mathematics*, sixth edition, January, 1949, was as follows: "The first English edition of EUCLID'S *Elements* was brought out at London in 1570, under the name of Sir HENRY BILLINGSLEY, later sheriff and Lord Mayor of London. (For the small part which BILLINGSLEY may have had to do with the translation see A. DEMORGAN, *British Almanac and Companion for 1837*, pp. 38–39 of *Companion*.) It contained 'a very fruitfull Praeface' by JOHN DEE, a man of great erudition; see W. F. SHENTON, 'The first English Euclid,' AM. MATH. MONTHLY, v. 35, 1928, pp. 505–512."

Not long after my *Outline* was published I discovered that this Note was mortifyingly inadequate in a work professing to present an up-to-date scholarly presentation of known facts in the field treated.

Happening to handle the *Mathematical Gazette*, v. 31, I found in the first number, Feb., 1947, that the frontispiece was a reproduction of the title-page of Billingsley's translation and in the note, "The first English Euclid," on the opposite page, the following sentences occur: "Perhaps because of Dee's notoriety, a good deal of credit for this translation has often been given to him, as for instance by DeMorgan, and also to an Augustinian friar, Whytehead, who appears to have been resident with Billingsley for a time. But, according to Rouse Ball, the copy of the Greek text of Theon's Euclid which Billingsley used, contains notes, comments and emendations in his own hand which make it clear that whatever may have been contributed by Dee and Whytehead the main work was done by Billingsley himself."

I then turned to W. W. R. BALL, *A History of the Study of Mathematics at Cambridge*, Cambridge, 1889, pp. 22–23, where I found the following: "In preparing this [English translation] he had the assistance both of Whytehead and of John Dee. In spite of their somewhat qualified disclaimers, it was formerly supposed that the credit of it was due to them rather than to him, especially as Whytehead, who had fallen into want, seems at the time when it was published to have been living in Billingsley's house. The copy of the Greek text of Theon's Euclid used by Billingsley has however been recently discovered, and is now in Princetown College, America;[1] and it would appear from this that the credit of the work is wholly due to Billingsley himself. The marginal notes are all in his writing, and contain comments on the edition of Adelhard and Campanus from the Arabic and conjectural emendations which shew that his classical scholarship was of a high order."

2. DeMorgan statements. Before indicating what are some of the principal statements of the Halsted Note let us quote portions of DeMorgan statements

[1] In the footnote there is a reference to G. B. HALSTED, "Note on the first English Euclid," *Amer. Jn. Math.*, v. 2, 1878, pp. 46–48.

in the *Companion*. "The principal work of Dee is the translation of Euclid, which appeared with his preface in 1573 [*sic*]. We call it his, though it is universally stated (and by himself* among others) to have been made by Sir Henry Billingsley. Considering that he wrote the preface, the notes, and the translation of the book of Mohammed of Badgad [*sic*] on the division of surfaces, which was published with the second edition; that he had lectured on Euclid in various places, and left behind him MSS. on the subject, and in particular 'Instructions and Annotations upon Euclid's Elements'; considering, also, that the name of Sir H. Billingsley is not mentioned in the preface to his own Euclid; we imagine that the translator was a pupil of Dee, who worked under general, if not special, instructions, and executed the more mechanical part of the undertaking. . . . This translation of Euclid was either made from the Greek, or corrected by the Greek, as is evident from comparing the early Latin versions and the Greek text with it. As there may be some dispute about the degree in which Greek was studied in England at the period of which we write, we shall annex one sentence of comparison, namely, the Greek text of the enunciation of the fourth proposition of the first book, the Latin of Campanus, and the English of Billingsley . . . Latin of Campanus (Editions of 1482, 1491, and 1516)."

Apart from the incorrect date given as the year of publication of Billingsley's Euclid, other inaccuracies may be pointed out.

DeMorgan states that Dee wrote "the Notes" for Billingsley's translation, whereas Dee's only notes are a few in connection with Books X–XIII of the *Elements*. Indeed, Dee in his autobiographical tracts distinctly states that, besides the introduction, he only contributed 'divers and many Annotations and Inventions mathematicall added in sundry places of the aforesaid English Euclide after the tenth booke of the same' (*Miscellanies of Chetham Soc.*, v. 1, no. 5, 1851, p. 73; also *Remains*, v. 24). The complete list of references for Dee's notes is folios 255 *verso*, 306, 311 *recto*, book X; 325 *verso*–326, 329 *verso*, 337 *recto*, 346–347, 348 *verso*, 349, 352, book XI; 356 *verso*–357, 359–362, 371, 376, 380, 381 *verso*, book XII; 391–394 *recto*, 395, 397 *verso*, book XIII.

DeMorgan states that Dee wrote "the translation of the book of Mohammed of Bagdad," whereas he merely published a Latin manuscript which he had in his library; see R. C. ARCHIBALD, *Euclid's Book on Division of Figures*, Cambridge, 1915.

DeMorgan writes "considering . . . that the name of Sir H. Billingsley is not mentioned in the preface to his own Euclid." Why should it be when his name as translator is on the title page and each of the immediately following pages of the preface is headed: "The Translator to the Reader"?

DeMorgan wrote: "As there may be some dispute about the degree in which Greek was studied in England at the period of which we write . . . , " and he then quotes from a Greek edition of Euclid of 1703, 133 years later!

DeMorgan refers to a second edition of Billingsley's work; it will presently appear that there was no such edition.

* In the list of his works, in his apologetical letter to the Archbishop of Canterbury.

DeMorgan states: "This translation [Billingsley's] was made from the Greek, or corrected by the Greek." It will presently appear that the translation was made from the Latin.

DeMorgan cites Campanus Latin editions of Euclid to fortify his argument. Since Billingsley used an entirely different Latin edition, based on translations from both the Greek and Arabic, the whole argument of DeMorgan falls to pieces.

3. Halsted. It was Professor Halsted who, 71 years ago, apparently first made known in print (at the reference given above) that the volume or volumes of which Billingsley made use, in preparing his English Euclid, were located in the library of Princeton University. Halsted concludes his article as follows: "By reading what he has done, it immediately appears that . . . the corrections he has actually made sufficiently prove his scholarship and render entirely unnecessary DeMorgan's suppositious aid from Dr. Dee, while, on the other hand, they establish the conclusion about the translation to which DeMorgan's sagacity had led him, that 'It was certainly made from the Greek, and not from any of the Arabico-Latin versions'."

"To the one sentence of comparison in proof of this published by DeMorgan, Billingsley's autograph indications would enable me to add as many as any one desired, but suffice it to say, that the definitions of the Eleventh Book are alone entirely decisive."

That these statements are wholly unreliable shall presently appear.

4. Sidney Lee. In Lee's biography of Billingsley in the *Dictionary of National Biography* (*DNB*) the following erroneous statement occurs: "His original was the Latin version attributed to Campanus, which had been first printed in 1482, and again in 1509."

5. John Aubrey. In Aubrey's famous *Brief Lives, edited from the Original Manuscripts and with an Introduction* by O. L. DICK, London, 1949, p. 89, we find in the biographical notes on John Dee, "His Picture in a wooden cutt is at the end of Billingsley's *Euclid*." This erroneous statement is also made in the *DNB* biography of Dee, 1888. Professor Shenton has well pointed out that this portrait is of John Day, the printer of Billingsley's volume. See J. JOHNSON, *Typographia*, v. 1, London, 1824, p. 540, where it is stated that, this woodcut bearing the date 1562, is a portrait of Day and "perhaps the earliest of an ancient printer, which can be depended upon as genuine." See also J. AMES, *Typographical Antiquities*, v. 1, London, 1785, p. 647.

6. The Billingsley volumes. The Librarian of Princeton University most graciously allowed these precious volumes to be sent to Brown University for my use.

Before describing one of these volumes it seems desirable to correct many prominent writers concerning the first Latin translation of Euclid's Elements, which was made from the Arabic, about 1120, by an Englishman, ADELHARD (or

Athelhard or Aethelhard) of Bath.† Many historians state that CAMPANO (usually called CAMPANUS) of Novarra, a mathematician, astronomer and medical man, made the first translation of Euclid's Elements from the Arabic and that his manuscript was published at Venice in 1482. This translation, made 150 years after that of Adelhard, was not independent of the first translation. The definitions, postulates and axioms, and the 364 enunciations are word for word identical in Adelhard and Campanus. . . . Campanus may have used Adelhard's translation and only developed the proofs by means of another redaction of the Arabian Euclid. . . . It seems most probable that Campanus stood to Adelhard somewhat in the relation of a commentator, altering and improving his translation by means of other Arabic originals. (Quoted from T. L. Heath, *A History of Greek Mathematics*, v. 1, Oxford, 1921, p. 362–364.) It is interesting that this same point of view is expressed in *Enciclopedia Italiana*, v. 1, 1929, article "Adelardo."

The first translation of Euclid's Elements from the Greek (a Theonine ms.) into Latin was made (after 7 years of labor) by BARTOLOMEO ZAMBERTI (b. 1473), and published at Venice in 1505. His animus against Adelhard-Campanus appears in a number of notes. The first printed edition of the Greek text was brought out at Basle (I. Hervagius, printer) by SIMON GRYNAEUS (d. 1541) in 1533. Combined with this 268-page work is a separately paged (115 pp.) PROCLUS commentary, in Greek, on the first book of Euclid's *Elements*.

The first of the editions giving Adelhard-Campanus's and Zamberti's translations in conjunction was brought out at Paris in 1516. (I now quote from Heath's edition of Euclid.) "The idea that only the enunciations were Euclid's, and that Campanus was the author of the proofs in his edition, while Theon was the author of the proofs in the Greek text, reappears in the title of this edition; and the enunciations of the added Books XIV, XV, are also attributed to Euclid, HYPSICLES being credited with the proofs. . . . The arrangement of the propositions is as follows: first the enunciation with the heading *Euclides ex Campano*, then the proof with the note *Campanus*, and after that *Campani additio*, any passage found in the edition of Athelhard-Campanus's translation but not in the Greek text; then follows the text of the enunciations translated from the Greek with the heading *Euclides ex Zamberto*, and lastly the proof headed *Theo ex Zamberto*. There are separate figures for the two proofs. This edition was reissued with few changes in 1537 and 1546 at Basel (*apud Iohannem Hervagium*), but with the addition of the *Phaenomena*, *Optica*, *Catoptrica*, *etc.*"

Of this 1546 edition there was (according to RICCARDI) an "unauthorized" edition of 1558. It is a copy of this volume, owned by Billingsley and filled with manuscript notes, which was basic in preparing the manuscript for his English Euclid.

† There is a brief sketch of Adelhard by R. ADAMSON in *Dict. Nat. Biog.*, v. 1, 1885. There is also the comparatively recent full-length biography: F. BLIEMETZRIEDER, *Adelhard von Bath. Blätter aus dem Leben eines englischen Naturphilosophen des 12. Jahrhunderts und Bahnbrechers einer Wiedererweckung* der griechischen Antike. *Eine kulturgeschichtliche Studie*. Munich, 1935, viii, 395 p.

The title of this volume is as follows: *Evclidis Mega-* | *rensis mathematici clarissimi* | *Elementorum geometrico-* | *rum libri XV.* | *Cum expositione Theonis in priores XIII à Bartholomaeo* | *Zamberto Veneto latinitate donata, Campani in* | *omnes, & Hypsiclis Alexandrini in du-* | *os postremos.* | *His adiecta sunt Phaenomena, Catoptrica & Optica,* | *deinde Protheoria Marini, & Data.* | *Postremùm uerò, Opusculum de Leui & Ponderoso,* | *hactenus non uisum, eiusdem autoris.* | [design reproduced in this article, in the illustration with the autograph, "Henricus Billingsley."] Cum priuilegio Caesareo. | Basileae, Per Ioannem Her- | uagium, & Bernhardum Brand, Anno | M.D. LVIII. It contains 537 folio pages. The six-page preface by Philip Melanchthon is dated Wittenberg, August, 1537.

Cum priuilegio Cæsareo.

BASILEAE, PER IOANNEM HER-
uagium, & Bernhardum Brand, Anno
M. D. LVIII.

There are many notes in margins evidently written by the person whose autograph is on the title page. At some time since the volume was originally used by Billingsley it has been rudely trimmed, probably when bound with another volume; in this way many of the manuscript notes are badly mutilated.

The other volume bound with the 1558 work is the 1533 Greek *editio princeps* to which we referred above. That this volume was also briefly used by Billingsley at some time is shown by slight manuscript notes in his handwriting, on pages 5–9, in connection with propositions 6–8, 13, 17, of the first book of the Elements. All the other 374 pages of the volume seem to be as clean as when they first came from the printer. The manuscript notes are of no importance. Hence further consideration of Billingsley's Greek edition of Euclid's *Elements* may be omitted. We shall presently report in some detail on his Latin edition.

7. Princeton and the 1558 volume. When or whence this most interesting volume became the property of Princeton University or Princeton College or the New Jersey College is unknown, but it would seem as if the Greek volume at least might have been in the College library for 135 years or more; for, on the fly leaf is: "Editio Princeps. Rare & Dear. P.L.," written doubtless by Dr. Philip Lindsley, Librarian in the College from 1812 to 1824. Hence since this Librarian knew the cost of the volume it may well have been acquired during his period of office. Lindsley's ms. Catalogue of books in the College Library (made in 1821) includes this volume. Assuming that the two Billingsley volumes were bound together at this time, and assuming further that the volume had been acquired even in 1812, Halsted's statement in 1879 that Princeton had the volume "for nearly a century, perhaps longer" is not by any means checked. The librarian of today has no such information. The Lindsley Catalogue makes no reference to the 1558 volume.

The present binding of the volume was put on in 1923. There is reason to surmise first, that the binding (of the two volumes in one, bound together previous to this date), had become so unsatisfactory, it was then replaced; secondly, that no trimming of the edges then occurred. The present accession number in the volume was given in 1892.

Whether or not Halsted was the first to discover (in print) the special interest of the volume because of Billingsley's signature is uncertain. On the inside cover of the new binding is pasted an anonymous undated, and unidentified periodical clipping, entitled "An archaeological treasure," which reveals the facts. The initials "J.T.D." are added in ink at the end of the clipping. It has been surmised that these initials probably refer to JOHN T. DUFFIELD (d. 1901) who was at Princeton a tutor in Greek (1845–47), and a member of the mathematics department (1847–98). One sentence of the article is: "It has been in the Library for at least half a century, but the fact that it at one time belonged to the first translator of Euclid into English does not seem to have been known until a few years ago."

8. The manuscript notes in the 1558 volume. The notes commence with the correction of a single Greek letter in Zamberti's discussion of book I, proposition 9 of the *Elements*, on page 14, and end with more than a dozen corrections of the text in proposition 9 of the *Phaenomena* on page 491. In the *Elements*, books

I–XV, there are notes on 256 pages, 21 of these being on the Adelhard-Campanus text, and 235 on material in Zamberti's Latin translation from the Greek. Hence there was a slight show of reason for scholars surmising that Billingsley's translation was from a Greek original, although in reality it was from a Latin translation of a Greek source.

The notes consist, practically wholly, of corrected misprints in letters, words, phrases, of deletions, phrase and word substitutions, additions of letter to figures,—such as every careful student would ordinarily make. All of these notes occur in connection with propositions. The only commentary on a definition is where, in book XI, the omission of a definition of a tetrahedron is noted; this makes the last sentence quoted above from Halsted appear as a decided mental lapse. Not a single note is concerned with setting forth original supplementary ideas. All of these were reserved for record in the so-called "translation," consideration of which, with reference to the Latin original, we shall now take up.

9. The 1558 and 1570 volumes. The title-page of Billingsley's work is reproduced in facsimile in Professor Shenton's interesting article referred to above [as well as in the *Mathematical Gazette*], and the Billingsley preface is quoted in full. The plan "to translate, and set abroad some other good authors, both pertaining to religion (as partly I have already done) and also pertaining to the Mathematicall Artes," referred to in the concluding lines of the preface, was never carried out, according to Lee, in *DNB*.

The preface is immediately followed by a large John Dee folding plate, of size 34.7×41.6 cm. in the Brown Univ. copy, "the Groundplat of my Mathematicall Preface"; this Preface occupies the next 46 pages. Then follow 464 folios 928 pages of a finely printed English edition of the *Elements* with beautifully designed initial letters throughout. The large D, at the beginning of Dee's "fruitfull Praeface" seems to contain his coat of arms.

The Billingsley volume is by no means merely a translation of parts of the 1558, but as the title page states: "Whereunto are annexed certaine Scholies, Annotations, and Inuentions, of the best Mathematiciens, both of time past, and in this our age." For example, the writings of the following authors are quoted: Apollonius, Archimedes, Barlaam, Boetius, Campanus, Dee, Eudoxus, Flussates, Hypsicles, Jordanus Nemorarius, Montaureus, Oenopides, Orontius, Pappus, Pelitarius, Plato, Proclus, Pythagoras, Regiomontanus, Sceubelius, Thales, Theodosius, Timeus. Thus in this single volume was brought together all the earlier and current commentary of importance. The print and appearance of the work was also worthy of its contents, and no pains were spared to represent everything in the clearest and most perfect form—a truly monumental work.

It is noticeable that the names of Theon and Zamberti are not included in the list of authors given above. But the explanation of this is that the Theon-Zamberti text was the one translated as fundamental—the one to which other

discussions were added. Thus the true Greek phrases, which DeMorgan noted, were preserved. Just this procedure would also be expected from the mass of manuscript corrections of the Theon-Zamberti text in the 1558 volume.

10. Authenticity of the Billingsley volumes. In order to check that the autograph and notes were actually in the handwriting of the former Lord Mayor of London, I procured from the British Museum photostat copies of two letters written by the Lord Mayor in 1587 and 1591 [Landsdowne Mss. 62, f. 41 and 67, f. 213]. The comparison of the notes with the writing of these letters was conclusive in showing that they were written by the same person.

11. Euclid Editions by Billingsley 1570, and Leeke & Serle 1661. In the passage by DeMorgan which we quoted above from the *Companion* he refers by indirection to the following work as a "second edition" of that by Billingsley: Euclid's | Elements | of | Geometry. | In XV. Books: | With a supplement of divers *Propositions* | and Corollaries. | To which is added, a *Treatise* of Regular Solids, | By *Campane* and *Flussas*. | Likewise | Euclid's Data: | And Marinus his Preface | thereto annexed. | *Also a* Treatise *of the* Divisions *of* Superficies *ascribed to* | Machomet Bagdedine, *but published* by Commandine, *at the* | *request of* John Dee *of* London; whose Preface *to the said* Treatise | *declares it to be the* Worke of Euclide, | *the* Author *of these* Elements. | Published by the Care and Industry of | John Leeke and George Serle, students | in the *Mathematicks*. | *London:* | Printed by R. & W. Leybourn, for George | Sawbridge at the *Bible* upon *Ludgate-hill*, *MDCLXI*. [44 p.+folding plate+6 p.]+650 p.+1 p., Errata.

Although this title-page refers to "XV Books" of the *Elements* the authors give "The sixteenth Element of Euclide," pages 490–522. This last page concludes with "*The End of the Sixteenth Element of* Euclide, *added by* Flussas." These books are practically identical in the two volumes. In each case this sixteenth book is followed by "A brief treatise added by Flussas" (folio verso 458–folio recto 463 in Billingsley, with which the volume concludes except for errata and colophon), (pp. 523–532 in Leeke & Serle).

The brief treatise of Flussas here quoted was from the edition of the *Elements* published at Paris in 1566, by Franciscus Flussates Candalla (François de Foix, Comte de Candale, 1502–1594). Two of the figures given here, showing how two Archimedean semiregular solids $P_2 \equiv (8_3, 6_4)$, $P_7 \equiv (20_3, 12_5)$, may be obtained by folding two pieces of appropriately cut paper are of special interest. On folios 320 *verso*–322 *verso* of Billingsley, similar figures are indicated for the five regular solids, but these do not appear in Leeke & Searle. Are these the earliest presentations of such figures? The answer to this question is in the negative. All of these figures and more of the Archimedean semi-regular solids were pictured by ALBRECHT DÜRER in his *Anderweysung der Messung mit dem Zirckel und richtscheyt in Linien, ebenen und gantzen Corporen*, Nürnberg, 1525, in the fourth book on regular bodies. See MAX STECK, *Dürers Gestaltlehre der Mathematik und der bildenden Künste*. Halle, 1948, pp. 64–74.

The first part of the Leeke & Serle volume (L. & S.) contains a dedication (2 p.); Dee's preface as in Billingsley (42 p.+folding plate); "To the Reader" (3 p.); "An account of the author according to ancient philosophers" (3 p.). Then begins the presentation of the Elements.

In the Billingsley volume (B.) each of the 16 books was preceded by an introduction and statement and discussion of definitions. In Books 2, 8, 9, L. & S. has no introductions or definitions and in the case of most of the other books these parts were appreciably shorter; in every case but for Book 12 the forms of statements were quite different. In exceptional manner L. & S. discussion of definitions in Book 1 extends to 17 p. as against 12 in B.

In Book 2, B. gave in connection with each proposition the numerical commentary of Barlaam the 14th century monk (later an abbot and a bishop) edited with Latin translation by Dasypodius in 1564. L. & S. put all of this Barlaam exemplification at the end of their Book 2.

The three kinds of lines in figures throughout L. & S. are explained as follows: "the greater black lines stand for, and denote the Data, or things given, the lesser black lines the things required, and the pricked lines serve for the Constructions and Demonstration; the like method is also observed in the Circular lines given and required." It is further stated in "To the Reader": "that the explications of the things given and required, are comprehended in the words of the Proposition, by means of the correspondent letters relating to the several parts of the Schemes, in such sort, as the Propositions may be read with them, or without them, and so by this means the Demonstrations are much abbreviated." Such editing is of course wholly foreign to Greek form.

The wording of proofs in L. & S. and B. are usually entirely different. L. & S. practically never give any secondary discussions of proofs.

Summing up, we may say that it is wholly improper to state, as DeMorgan does, that the L. & S. volume is a second edition of B. It is simply an independent, and inferior, edition of the *Elements* in the preparation of which L. &. S.—the obscure "students in Mathematicks"—perhaps consulted B. constantly.

The remaining two works of L. & S., not in B. are the first English editions of: (a) Euclid's *Data* (pages 533–600, pages 533–539 being "A commentary or Preface written by the Philosopher Marinus); and (b) *A Book of Divisions of Superficies*, ascribed to Machomet Bagdedine. This book, pp. 601–650, has a special title-page dated 1660. This work was translated from the Latin edition of 1570 published by Dee & Commandinus.

Two other English editions of the *Data* were in editions of Euclid's Elements published by ROBERT SIMSON (1687–1768), Glasgow, 1756, and by THOMAS HASELDEN, in an edition of ISAAC BARROW (1630–1677), London, 1732. Of each of these works there were many other editions.

To the list of Corrigenda for my *Outline of the History of Mathematics*, sixth edition, noted in this MONTHLY, v. 56, Aug.–Sept. 1949, the following may now be added:

P. 13, line 22, *for* 2β as, *read* an

P. 13, line 23, *for* Diophantine equation, *read* Diophantine quadratic equation

P. 33, line 7, for CAMPANUS, read ADELHARD-CAMPANUS

P. 38, line 28, *add:* And in his *Traité du Triangle Arithmetique*, 1665, is a good statement of the method of mathematical induction

P. 59, line 28, *for* 303, *read* 302

P. 61, note 46b, line 3, *for* 1932, *read* 1944, and *for* TAER, *read* THAER

P. 64, note 75, line 2, *for* earliest mathematical, *read* earliest Greek mathematical

P. 71, note 123, line 4, *interchange the values of x and y*

P. 82, note 201, line 9, *for* Levi S. Smith, *read* Levi B. Smith

P. 88, line 4, for MALMKE, *read* MAHNKE

P. 90, note 238, line 3, *delete* Z.d.

P. 94, note 259, *for* 1948, *read* 1918

P. 98, line 1, *for Suppl. Revue, read Suppl. Elemente d. Mathematik*, or *Revue*

P. 104, *after* Adelard of Bath (12th cent.), *add* 33,

P. 107, *for* Erastosthenes, *read* Eratosthenes

P. 110, *for* Malmke, *read* Mahnke, *and put after* Magowan

P. 113, *for* Smith, L. S., *read* Smith, L. B., and *for* TAER [**46b**], *read* THAER [**46, b**] *and put after* Terquem.

In the Index the following dates of death may be added: Engebach, 1946; Nunn, 1944; Ruska, 1949; Simons, 1949; J. W. A. Young, 1948.

To note 208 the following paragraph may be added:

In June 1915 Dr. GEORGE A. PLIMPTON presented to, and installed in, the Delta Kappa Epsilon House at Amherst College a portrait of Sir Isaac Newton, and a fireplace made of wood and bricks taken from the Newton residence referred to above as supplying the Babson Institute "foreparlor." The Amherst Newtoniana are now in the library of the Fraternity House, and a view of them is given on the last page of *Memorial Rooms at Amherst College*, 1937.

SOME NUMERICAL METHODS FOR SOLVING SYSTEMS OF LINEAR EQUATIONS

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Recent papers have exhibited the basic identity of some apparently distinct numerical methods of solving linear systems (see the list of papers at the end). It seems worth-while to note such basic identities since the single general principle under which the methods can be subsumed may well suggest others, and while one may agree with von Neumann and Goldstine [10], Bodewig [1], and others that the Gaussian method of elimination is the best direct method in general, special circumstances will often indicate special methods. We shall therefore attempt to summarize and unify some of these methods.

1. Direct methods. Laderman [7] and Dwyer [3] recommend the use of the "Square-rooting" method of Choleski for symmetric matrices. Bodewig [1] states, von Neumann and Goldstine [10] imply, that this is in reality identical with the Gaussian method of elimination. If the equations are, in matrix form,

$$Ax = y, \tag{1}$$

and A is symmetric, one seeks a triangular matrix L such that

$$A = L'L, \tag{2}$$

the prime denoting transpose. The columns of L are determined sequentially so that in the ith column at most the first i components are non-null. These are chosen so as to give the correct scalar products with the previous i-1 vectors and the correct length, as fixed by the first i components of the ith column of A. In this form square roots occur, but by an obvious modification one can replace this decomposition by

$$A = M'D^2M, \tag{3}$$

where M is triangular with units along the diagonal, while D^2 is a diagonal matrix. This is the form achieved by von Neumann and Goldstine by Gaussian elimination. The point of the decomposition is, of course, the ready invertibility of triangular and of diagonal matrices.

Fox, Huskey and Wilkinson [4] describe a method of orthogonal vectors which they assert to be identical with the "escalator" method of Morris. Their method is the following: One seeks a set of vectors x_i, which are mutually orthogonal with respect to A as a metric tensor. This is to say that if X is the matrix whose columns are the x_i, then

$$X'AX = D^2, \tag{4}$$

where D^2 is a diagonal matrix. One then expresses the required vector x as a linear combination of the columns of X; that is,

$$x = Xw. \tag{5}$$

This is readily done once X is found, since the Equations (1) have clearly become

$$AXw = y, \tag{6}$$

and by (4)

$$D^2w = X'y. \tag{7}$$

In fact,

$$w = D^{-2}X'y$$

and therefore, by (5),

$$x = XD^{-2}X'y. \tag{8}$$

Now the orthogonal vectors x_i can be chosen by orthogonalizing any set of independent vectors, but a natural choice is the set of coordinate vectors. If these vectors are orthogonalized in sequence, then the matrix X is triangular, and, since it is clear that the matrix product in (8) multiplying y is A^{-1}, the relation of this method to those of Gauss and Choleski is exhibited.

Bodewig [1] attributes to Schmidt what amounts to an adaptation of Choleski's method to equations with a non-symmetric matrix, and Turing [9] has also made such an adaptation. The method of orthogonal vectors can be extended immediately if one orthogonalizes, in the ordinary sense, the column vectors of the matrix A. Thus if these column vectors are $a_1, a_2, \cdots$, one chooses a_1 to be the first vector, one replaces a_2 by a combination of a_1 and a_2 that is orthogonal to a_1, and so on. These multipliers form a triangular matrix X whose diagonal elements are all units, and such that

$$(AX)'(AX) = D^2. \tag{9}$$

One uses (5) again so that the equations become

$$AXw = y, \tag{10}$$

whence

$$\begin{aligned} D^2w &= X'A'y, \\ w &= D^{-2}X'A'y, \\ x &= XD^{-2}X'A'y. \end{aligned} \tag{11}$$

Note that X is triangular, and that $X'A'$ is the transpose of the matrix of orthogonalized vectors.

This is, of course, the result one would arrive at from solving

$$A'Ax = A'y \tag{12}$$

by the method of orthogonal vectors, except that the actual formation of the matrix product $A'A$ is unnecessary. Moreover, the normal equations in correla-

tion theory are of the form (12) in which generally A is a rectangular matrix having more rows than columns. Approached in this fashion the equations can be solved without having to be written explicitly.

The method of orthogonal vectors is, in a sense, the inverse of the method of Choleski, since one factorizes A^{-1} rather than A, and the foregoing extension to non-symmetric matrices is the extension of the method of orthogonal vectors. In the Schmidt extension of Choleski's method as described by Bodewig, one orthogonalizes the rows, rather than the columns of A, so that the triangular matrix X is such that

$$XAA'X' = D^2. \tag{13}$$

If

$$A'X'w = x$$

so that

$$AA'X'w = y$$

then

$$D^2w = Xy$$

and

$$x = A'X'D^{-2}Xy. \tag{14}$$

2. Successive approximations. If the matrix A is positive definite we may adopt it as the metric tensor. The equations (1) may be interpreted as relating the contravariant representation x to the covariant representation y of the same vector, the latter being given, the former required. Suppose $x^{(p)}$ is some approximation to x. One seeks an approximation $x^{(p+1)}$ which is to be in some sense better and so that the sequence approaches x as a limit. The vector whose contravariant representation is $x^{(p)}$ has $Ax^{(p)}$ as its covariant representation, and it differs from the required covariant representation by an amount

$$r^{(p)} = y - Ax^{(p)}. \tag{15}$$

This we may call the covariant representation of the pth residual. If u is the contravariant representation of any vector whatever that is not parallel to the residual, then we may project the residual orthogonally on this arbitrary vector, adjoin the projection to $x^{(p)}$ to get $x^{(p+1)}$, and have remaining a residual $r^{(p+1)}$ that is surely smaller than $r^{(p)}$.

To do this one forms $x^{(p)}+\lambda u$ and asks that λ be such that u is orthogonal to the difference between the vectors whose representations are λu and r:

$$u'(\lambda Au - r^{(p)}) = 0$$

$$\lambda = u'r^{(p)}/u'Au.$$

For uniformity we shall write, then,

$$\lambda^{(p)} = u^{(p)\prime} r^{(p)} / u^{(p)\prime} A u^{(p)}, \tag{16}$$

$$x^{(p+1)} = x^{(p)} + \lambda^{(p)} u^{(p)}, \tag{17}$$

and the sequence so defined will converge to x unless, perhaps, one happened to make some peculiar choice of the sequence $u^{(p)}$.

In principle any prescription for choosing a sequence of vectors u would lead to a method of successive approximations. In the method of steepest descent (8) one chooses

$$u^{(p)} = r^{(p)}. \tag{18}$$

In a sense this choice is optional. One would like, of course, to take the vector whose covariant representation is $r^{(p)}$, but one needs the contravariant representation of that vector, and finding this is equivalent to solving the equation. Hence one uses instead the vector whose contravariant representation is $r^{(p)}$. The more nearly orthogonal the base vectors, the more perfect is the choice. But of course if the vectors were orthogonal the matrix A would be diagonal and the equations no trouble in the first place.

In the Gauss-Seidel iteration, one chooses for the vectors u the coordinate vectors taken in rotation, from the first through the last and back to the first. In the method of relaxation one again chooses the coordinate vectors but not in any specified order. One examines at each stage the components of $r^{(p)}$, and if the kth component is largest, one chooses the kth coordinate vector for $u^{(p)}$. One could think of many other possibilities, but probably none so easy to apply.

Adaptation of this basic idea to non-symmetric matrices is almost immediate. One now uses the identity matrix as the metric tensor and considers the space of row vectors of A, or the space of column vectors. Bodewig describes two methods, one due to Kaczmarz and one to Cimmino, both using the space of row vectors. The equations then represent hyperplanes, and one seeks their intersection.

By the method of Kaczmarz, one passes from any approximation $x^{(p)}$ to $x^{(p+1)}$ by projecting the point $x^{(p)}$ orthogonally on one of the hyperplanes. Actually the projection can be made on any hyperplane through x, *i.e.* on a plane

$$u'Ax = u'y \tag{19}$$

where u is an arbitrary column vector. Since $u'A$ is normal to the plane, the projection is

$$x^{(p)} + \lambda A'u = x^{(p+1)}, \tag{20}$$

where λ is so chosen that $x^{(p+1)}$ lies in the plane (19):

$$u'A(x^{(p)} + \lambda A'u) = u'y$$

$$\lambda u'AA'u = u'y - u'Ax^{(p)} = u'r^{(p)}.$$

Hence

$$\lambda = u'r^{(p)}/u'AA'u. \tag{21}$$

If u is one of the coordinate vectors, λ is one component of the residual divided by the sum of squares of the coefficients of the corresponding equation. These sums of squares could all be calculated in advance and each step in the sequence should be fairly simple.

The method of Cimmino consists in reflecting $x^{(p)}$ into each hyperplane. Then $x^{(p)}$ and its n reflections all terminate on a single hypersphere about x, and the centroid of these points, with any associated set of non-negative weights, lies within this hypersphere and defines $x^{(p+1)}$. The method of Kaczmarz (as Bodewig remarks) becomes a special case of this when two weights only are taken non-null and equal.

A. de la Garza has described a method based upon the space of column vectors of A. In this picture, the equations to be solved required the resolution of the vector y along the n column vectors of A, and the components of x are the required multipliers. Any approximate set of multipliers $x^{(p)}$ will yield a vector $Ax^{(p)}$ deviating from y by

$$r^{(p)} = y - Ax^{(p)}.$$

Then if u is an arbitrary set of multipliers, one seeks an optimal improvement of the form λu:

$$x^{(p+1)} = x^{(p)} + \lambda u.$$

The optimal improvement is, of course, the orthogonal projection of $r^{(p)}$ on the vector Au:

$$u'A'(r^{(p)} - \lambda Au) = 0,$$

$$\lambda = r^{(p)\prime}Au/u'A'Au. \tag{22}$$

Again the simplest choice for u is a coordinate vector, in which case the denominator is now a sum of squares of a column. This method has the disadvantage of bringing the matrix A into the numerator for λ. It has, however, the decided advantage of requiring the adjustment of only one component of $x^{(p)}$, when u is a coordinate vector. In Kaczmarz's method, it is to be noted, every component of $x^{(p)}$ is adjusted by an amount proportional to its coefficient in one of the equations. The multiplier λ provides the constant of proportionality.

3. Inversion of matrices. In principle y could be a matrix of any number of columns and x a matrix of an equal number of columns, and the foregoing methods could be applied to the simultaneous solution of all the sets represented. In equation (16), for example, $\lambda^{(p)}$ would be a row vector to multiply on the right of $u^{(p)}$. Hence y could be the identity matrix, in particular, and x is then A^{-1}. In practice, however, it is probably better to observe the following: In the Gauss-Seidel method, the adjustment at any stage consists in solving one of the equa-

tions exactly for the corresponding x component, using the current estimates for the other components. The result of a complete cycle of such operations, passing from the first to the last of the equations in order, can be expressed in the form

$$A_1 x^{(p+n)} = y - A_2 x^{(p)}, \tag{23}$$

where

$$A = A_1 + A_2, \tag{24}$$

A_2 being the triangular matrix of supra-diagonal elements, A_1 the matrix of diagonal and subdiagonal elements. Then A_1 is triangular, non-singular, and readily inverted. Hence it is probably advantageous to write

$$x^{(p+n)} = A_1^{-1} y - A_1^{-1} A_2 x^{(p)} \tag{25}$$

whenever y is a matrix of several columns, and, more particularly, whenever $y = I$ and A^{-1} is required. Applied to this case, if $C^{(p)}$ is any approximation to A^{-1}, one can now write the sequence

$$C^{(r+1)} = A_1^{-1}(I - A_2 C^{(r)}). \tag{26}$$

It is immediately suggested that other decompositions such as (24) might be made, where one would require only that A_1 be easily inverted. In fact, one method that is used sometimes, to advantage when IBM equipment is available, is to take for A_1 the matrix of diagonal elements of A_1, A_2 containing the off-diagonal elements. This amounts to making a complete cycle as in Gauss-Seidel except that the corrections are not incorporated until the cycle is complete.

We shall not follow this line any farther since the geometry tends to get lost in the analysis. Reference should be made to Hotelling [5, 6], and to Bodewig [1] who rediscovers some of Hotelling's results.

4. Conclusion. This does not exhaust the list of either direct methods or methods of successive approximation. In particular Bodewig [1] describes several methods of the latter type for which the geometry is not apparent. Moreover, the fact that two methods may be demonstrably identical in principle does not necessitate that the sequence of numerical operations will be the same, and von Neumann and Goldstine [10] have emphasized the non-commutativity of some of the "pseudo-operations." Nevertheless, an appreciation of the significance of the arithmetic operations should be of assistance in adapting a method to the individuality of the problem in hand.

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THE GENERALIZED LEIBNIZ FORMULA

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1. Introduction. In 1695 and 1696 Leibniz and Johannes Bernoulli carried on a correspondence in the course of which they agreed on the symbol $\int$ for the integral and on d^n for the differential of the nth order. This latter symbol appeared in the famous letter of Leibniz to Bernoulli dated October, 1695, in which Leibniz pointed out the analogy between powers of a sum and differentials of a product, and stated the formula now known as the Leibniz formula for the nth derivative of the product of two functions. He also pointed out that integrals of the nth order can be regarded as differentials of the $(-n)$th order [1].

A little later in a letter to Wallis [2] dated May 28, 1697 Leibniz again briefly mentioned this "marvelous analogy" (*Et notavi mirabilem analogiam relationis inter differentias et summas*) but did not there carry the idea further. It was not until several years later that the formula was first printed in the publications of the Berlin Academy [3].

In the present note the formula is extended to differential operators with constant coefficients, both direct and inverse, and given a noninductive proof which in a sense explains Leibniz' analogy. Various applications are developed, including a systematic foundation of the theory of linear differential equations with constant coefficients.

2. Direct operators. Let $D^n u$ indicate the nth derivative with respect to x of any sufficiently differentiable function, $u(x)$. If then an operator of the form,

$$a_i D + b_i, \qquad i = 1, 2, \cdots, n, \ a_i, \ b_i \text{ independent of } x, \tag{1}$$

is applied to u and then another operator of this form is applied to the result, and so on, it is easily established that the final result is the same as if these operators had been formally multiplied and the result applied to u. In fact, these operators obey the associative and distributive and exponential laws of algebra, and may be added and multiplied exactly as if they were algebraic quantities.

Similarly, if $u(s)$ and $v(t)$ are respectively sufficiently differentiable functions of the two independent variables s and t, we may indicate partial derivatives of the product uv by $D_1^n uv = \partial^n uv/\partial s^n$, and $D_2^n uv = \partial^n uv/\partial t^n$. Then if operators of the form,

$$a_i D_1 + b_i D_2 + c_i, \qquad i = 1, 2, \cdots, n, \ a_i, \ b_i, \ c_i \text{ independent of } s \text{ and } t, \tag{2}$$

are, as above, successively applied to uv, they obey the same laws of algebra and may be added and multiplied exactly as if they were algebraic quantities. In fact, since u and its derivatives are independent of t, while v and its derivatives are independent of s, this statement is a consequence of the former.

3. Generalized Leibniz formula for direct operators. Let $f(D)$ be an integral rational function of the above operator D of the degree n, which is to be applied to the product $u(x)v(x)$ of two functions of x. We shall do this by first regarding u and v as respective functions $u(s)$ and $v(t)$ of two independent variables, s and t, and then later make the change of variables identifying s and t with x; $s = t = x$. With the above notation and this change of variables, we have

$$Duv = vDu + uDv = (D_1 + D_2)uv$$

and with the aid of the principles of §2 we may write

$$f(D)uv = f(D_1 + D_2)uv = f\{(D_1 - a) + (D_2 + a)\}uv, \quad a \text{ independent of } x. \tag{3}$$

We now apply the algebraic identity,

$$f(h + k) = f(k) + \frac{h}{1!}f'(k) + \frac{h^2}{2!}f''(k) + \cdots + \frac{h^n}{n!}f^{(n)}(k),$$

to the operator $f\{(D_1 - a) + (D_2 + a)\}$, obtaining

$$f(D)uv = \left\{f(D_2 + a) + \frac{D_1 - a}{1!}f'(D_2 + a) + \frac{(D_1 - a)^2}{2!}f''(D_2 + a) + \cdots + \frac{(D_1 - a)^n}{n!}f^{(n)}(D_2 + a)\right\}uv, \tag{4}$$

and on the setting $s = t = x$ in the right member, we find that

$$f(D)uv = u \cdot f(D + a)v + \frac{D - a}{1!}u \cdot f'(D + a)v + \frac{(D - a)^2}{2!}u \cdot f''(D + a)v + \cdots + \frac{(D - a)^n}{n!}u \cdot f^{(n)}(D + a)v, \tag{5}$$

which is the Generalized Leibniz Formula for Direct Operators.

For $f(D)=D^n$ and $a=0$ this becomes the usual form of the Leibniz formula for the nth derivative of a product, that is,

$$D^n uv = u\cdot D^n v + nDu\cdot D^{n-1}v + \frac{n(n-1)}{2!}D^2u\cdot D^{n-2}v + \cdots + D^n u\cdot v,$$

the coefficients being those of the binomial theorem.

4. Special properties of direct operators. Equation (5) is especially useful in certain cases. Thus if $u=e^{ax}$, then $(D-a)u=0$, and the entire second member of equation (5) vanishes except the first term. We thus obtain for direct operators the important Formula for Exponential Shift, namely,

$$f(D)e^{ax}v = e^{ax}f(D+a)v. \tag{6}$$

By expressing $\cos \omega x$ and $\sin \omega x$ in exponential form and applying the above formula we may derive the analogous Formula for Trigonometric Rotation,

$$\begin{aligned} f(D)\cos \omega x\cdot v &= \cos \omega x\cdot f_1(D)v - \sin \omega x\cdot f_2(D)v, \\ f(D)\sin \omega x\cdot v &= \sin \omega x\cdot f_1(D)v + \cos \omega x\cdot f_2(D)v, \end{aligned} \tag{7}$$

where f_1 and f_2 are defined by the identity, $f(D+\omega i)=f_1(D)+if_2(D)$.

If $u=e^{ax}P(x)$, in which $P(x)$ is an integral rational function of x of degree p, by application of formula (6) and then formula (5) we have,

$$f(D)e^{ax}P(x)v(x) = e^{ax}f(D+a)P(x)v(x),$$

$$f(D)e^{ax}P(x)v(x) = e^{ax}\left\{Pf(D+a)v + \frac{P'}{1!}f'(D+a)v + \cdots + \frac{P^{(p)}}{p!}f^{(p)}(D+a)v\right\}. \tag{8}$$

For $v=1$ this reduces to,

$$f(D)e^{ax}P(x) = e^{ax}\left\{f(a)P + \frac{f'(a)}{1!}P' + \frac{f''(a)}{2!}P'' + \cdots + \frac{f^{(p)}(a)}{p!}P^{(p)}\right\}$$

and for $a=0$ and $P(x)=x^p$, we find

$$f(D)x^p v(x) = x^p f(D)v + px^{p-1}f'(D)v + \frac{p(p-1)}{2!}x^{p-2}f''(D)v + \cdots + f^{(p)}(D)v,$$

the coefficients being those of the binomial theorem.

Evidently formula (8) may be extended to $f(D)\cos \omega x\ P(x)v(x)$ and to $f(D)\sin \omega x\ P(x)v(x)$ by the procedure used in developing equations (7).

5. Application to homogeneous linear equations with constant coefficients. The general solution of the homogeneous linear differential equation with constant coefficients,

$$f(D)y = 0, \tag{9}$$

is easily deduced from equation (5). Let r be a root of multiplicity m of the characteristic equation, $f(r)=0$. Then $f(D)$, $f'(D)$, $\cdots$, $f^{(m-1)}(D)$ all have the factor $D-r$ and yield zero when operating on e^{rx}. Also if $P(x)$ is an arbitrary integral rational function of x of degree $m-1$,

$$P(x) = C_0 + C_1x + C_2x^2 + \cdots + C_{m-1}x^{m-1},$$

then D^m, D^{m+1}, $\cdots$, D^n all yield zero when operationg on $P(x)$. Thus in equation (5) with $a=0$, if we take $v=e^{rx}$, the first m terms on the right vanish; and if we take $u=P(x)$, the remaining terms likewise vanish and, hence,

$$y = P(x)e^{rx} \tag{10}$$

is a solution of equation (9) involving m independent arbitrary constants. The sum of the solutions thus obtained from the various roots of the characteristic equation will thus involve a number of independent arbitrary constants equal to the sum of the multiplicities of the roots of the characteristic equation and hence equal to the order of the given equation (9). It is thus the general solution.

6. Inverse operators. If $f(D)$ is an integral rational function of D, then we agree to indicate by

$$y(x) = \frac{1}{f(D)}u(x) \tag{11}$$

any function $y(x)$ satisfying the equation,

$$f(D)y(x) = u(x). \tag{12}$$

The function $y(x)$ indicated by the inverse operation $\{1/f(D)\}\ u(x)$ is thus not unique, but the difference between any two possible determinations is a solution of the corresponding homogeneous equation, $f(D)y(x)=0$. A statement that two inverse operators are equal is admissable in solving equation (12) if the statement is reducible to an identity by the application of $f(D)$ to both members. It follows that inverse operators in D may be transformed exactly as if they were algebraic quantities, since the direct operators are known to possess this property.

However, when an inverse operator, not a factor of $1/f(D)$, occurs in an algebraic transformation of $1/f(D)$, it is necessary that the same determination of this inverse operation be employed wherever it occurs, as otherwise there is no assurance that these terms would cancel properly if $f(D)$ were applied. This obvious requirement is often overlooked [4].

With the definitions and notation of §2 we may, as above, define

$$y(s, t) = \frac{1}{f(D_1, D_2)} u(s)v(t) \tag{13}$$

as any solution of the differential equation,

$$f(D_1, D_2)y(s, t) = u(s)v(t) \tag{14}$$

in which $f(D_1, D_2)$ is an integral rational function of D_1 and D_2, and s and t are two independent variables. Since $u(s)$ and its derivatives and integrals play the role of constant factors in operations with D_2, while $v(t)$ and its derivatives and integrals play a like role in operations with D_1, it follows that the conclusions of the above paragraphs also apply here, and the operator $1/f(D_1, D_2)$ may be modified in form by algebraic operations subject to the precaution mentioned above.

7. Generalized Leibniz theorem for inverse operators. Now let $1/f(D)$ be an inverse operator which is to be applied to the product $u(x)v(x)$ of two functions of x. As in §3, we first regard u and v as respective functions $u(s)$ and $v(t)$ of two independent variables s and t, and then after any desired algebraic transformations, set $s=t=x$. For to prove that

$$\frac{1}{f(D)} u(x)v(x) = \frac{1}{f(D_1 + D_2)} u(s)v(t), \tag{15}$$

we have merely to observe that the two members yield the same result when operated upon by the respective direct operators, $f(D)$ and $f(D_1+D_2)$, which are equivalent under this change of variables.

For greater generality let us consider the operation,

$$F(D)u(x)v(x) = \frac{P(D)}{Q(D)} u(x)v(x),$$

in which $P(D)$ and $Q(D)$ are integral rational functions of D of degrees p and q, respectively. We introduce the variables s and t, as above, and write

$$F(D)u(x)v(x) = \frac{P(D_1 + D_2)}{Q(D_1 + D_2)} u(s)v(t).$$

We expand P and Q in successive powers of D_1-a, a constant, as in equation (4),

$$P(D_1 + D_2) = P(D_2 + a) + \frac{D_1 - a}{1!} P'(D_2 + a) + \frac{(D_1 - a)^2}{2!} P''(D_2 + a) + \cdots + \frac{(D_1 - a)^p}{p!} P^{(p)}(D_2 + a)$$

$$Q(D_1 + D_2) = Q(D_2 + a) + \frac{D_1 - a}{1!} Q'(D_2 + a) + \frac{(D_1 - a)^2}{2!} Q''(D_2 + a) + \cdots + \frac{(D_1 - a)^q}{q!} Q^{(q)}(D_2 + a)$$

and perform the indicated division, obtaining the algebraic identity

$$F(D) = \frac{P(D_1 + D_2)}{Q(D_1 + D_2)} = A_0(D_2 + a) + \frac{D_1 - a}{1!} A_1(D_2 + a) + \frac{(D_1 - a)^2}{2!} A_2(D_2 + a) + \cdots + \frac{(D_1 - a)^{n-1}}{(n-1)!} A_{n-1}(D_2 + a) + \frac{(D_1 - a)^n}{n!} \frac{R_n(D_1, D_2)}{Q(D_1 + D_2)},$$

where

$$A_0 = \frac{P}{Q}, \qquad A_1 = \frac{P'Q - PQ'}{Q^2}, \cdots,$$

and in which R_n involves $D_1 - a$ only in positive integral powers of degree not greater than the larger of the two numbers $p-n$ and $q-1$. Regarded as an algebraic identity in D_1, the last or remainder term on the right vanishes together with its first $n-1$ derivatives for $D_1 = a$, and consequently the polynomial of the $(n-1)$th degree in D_1, represented by the other terms on the right, must possess the same value and first $n-1$ derivatives as $F(D_1+D_2)$ for $D_1 = a$. Thus,

$$A_i(D_2 + a) = F^{(i)}(D_2 + a), \quad i = 0, 1, 2, \cdots, n - 1$$

and on setting $s = t = x$, we have the Generalized Leibniz Theorem for Inverse Operators, that is,

$$F(D)uv = u \cdot F(D + a)v + \frac{D - a}{1!} u \cdot F'(D + a)v + \frac{(D - a)^2}{2!} u \cdot F''(D + a)v + \cdots + \frac{(D - a)^{n-1}}{(n-1)!} u \cdot F^{n-1}(D + a)v + \left\{\frac{(D_1 - a)^n R_n(D_1, D_2)}{n! Q(D)}\right\} uv. \tag{16}$$

8. Special properties of inverse operators. The remainder term in equation (16) generally becomes complicated for large values of n, but in certain important cases it takes a simple form. Thus if $u=e^{ax}$, then $(D-a)u=0$ and the entire second member of equation (16) vanishes with the exception of the first term. Formulas (6), (7) and (8) previously proven for direct operators now follow at once for inverse operators, $F(D)$, by the same argument as before.

In the special case, $F(D)=1/D$, $a=0$, of equation (16) we have, $R_n=n!/(-D_2)^n$ and the equation yields the familiar Formula for Iterated Integration by Parts, namely,

$$\frac{1}{D}uv = u\cdot\frac{1}{D}v - Du\cdot\frac{1}{D^2}v + D^2u\cdot\frac{1}{D^3}v + \cdots + (-D)^{n-1}u\cdot\frac{1}{D^n}v + \frac{1}{D}\left\{(-D)^n u\cdot\frac{1}{D^n}v\right\}. \tag{17}$$

As one more special case we might mention an analogous formula for double iterated integration by parts, similarly obtained,

$$\frac{1}{D^2}uv = u\cdot\frac{1}{D^2}v - 2Du\cdot\frac{1}{D^3}v + 3D^2u\cdot\frac{1}{D^4}v + \cdots + n(-D)^{n-1}u\cdot\frac{1}{D^{n+1}}v + \frac{(-1)^n}{D^2}\left\{(n+1)D^n u\cdot\frac{1}{D^{n+2}}v + nD^{n+1}u\cdot\frac{1}{D^{n+3}}v\right\}.$$

If in equation (17) we take $u=(b-x)^n/n!$, then u and all its derivatives except the nth vanish for $x=b$, and $(-D)^n u=1$. And if we choose for $(1/D^m)v$, $m=1, 2, \cdots, n+1$, that determination vanishing for $x=a$, then all the terms in the second member of equation (17) except the last vanish both for $x=a$ and $x=b$, and we have

$$\frac{1}{D}\frac{(b-x)^n}{n!}v(x)\Big]_a^b = \frac{1}{D^{n+1}}v(x)\Big]_a^b. \tag{18}$$

With a slight change in notation and exchange of members, this is the well known Reduction Formula,

$$\int_a^b\int_a^{x_1}\int_a^{x_2}\cdots\int_a^{x_n} f(x_{n+1})dx_{n+1}dx_n\cdots dx_1 = \int_a^b \frac{(b-x)^n}{n!}f(x)dx.$$

9. Application to nonhomogeneous linear equations with constant coefficients. A particular integral of the special differential equation,

$$f(D)y = e^{ax}P(x),$$

in which $P(x)$ is an integral rational function of x of degree p, is readily obtained from the generalized Leibniz theorem for inverse operators. Assuming first that

$f(a) \neq 0$, we have merely to write equation (8) for the inverse operator $F(D) = 1/f(D)$ in the special case $v \equiv 1$ developed in §4,

$$(19)\quad F(D)e^{ax}P(x) = e^{ax}\left\{F(a)P + \frac{F'(a)}{1!}P' + \frac{F''(a)}{2!}P'' + \cdots + \frac{F^{(p)}(a)}{p!}P^{(p)}\right\},$$

and the desired integral is obtained.

If, however, $f(D)$ is of the form, $f(D) = (D-a)^m\phi(D)$, $\phi(a) \neq 0$, then we have

$$F(D) = \frac{1}{f(D)} = \frac{1}{(D-a)^m}\Phi(D), \quad \text{where} \quad \Phi(D) = \frac{1}{\phi(D)}.$$

Hence by property (6),

$$F(D)e^{ax}P(x) = \Phi(D)e^{ax}\frac{1}{D^m}P(x) = \Phi(D)e^{ax}P^{(-m)},$$

in which negative superscripts on P indicate integration. We have now merely to write equation (19) with Φ for F and $P^{(-m)}$ for P,

$$(20)\quad F(D)e^{ax}P(x) = e^{ax}\left\{\Phi(a)P^{(-m)} + \frac{\Phi'(a)}{1!}P^{(1-m)} + \frac{\Phi''(a)}{2!}P^{(2-m)} + \cdots + \frac{\Phi^{(p)}(a)}{p!}P^{(p-m)}\right\},$$

and the desired integral is obtained. Evidently equation (19) is merely the special case of equation (20) for $m = 0$. In applying equation (20), any individual terms of lower degree than m may of course be dropped, as they will appear in the complementary function.

Evidently the above solution may be extended to equations of the type,

$$f(D)y = \cos \omega x \cdot P(x), \qquad f(D)y = \sin \omega x \cdot P(x)$$

by the device employed in developing equation (7).

References

1. C. J. Gerhardt, Leibnizens mathematische Schriften, Vol. III, p. 221.
2. Commercium Epistolicum, No. LXXVI, Edition of Biot and Lefort, p. 165.
3. C. J. Gerhardt, Leibnizens mathematische Schriften, Vol. III, p. 228.
4. W. W. Johnson, On a Point Connected with Symbolic Methods of Integration, The Messenger of Mathematics, Vol. 16, p. 86.

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

L. E. BUSH,* College of St. Thomas

The following results of the tenth annual William Lowell Putnam Mathematical Competition, held March 25, 1950, have been determined in accordance with the constitution of the Competition. This Competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of The Mathematical Association of America.

The first prize, four hundred dollars, is awarded to the Department of Mathematics of California Institute of Technology, Pasadena, California. The members of the team were J. Brody, H. A. Forrester, R. S. Pierce; to each of these a prize of forty dollars is awarded.

The second prize, three hundred dollars, is awarded to the Department of Mathematics of Harvard College, Cambridge, Massachusetts. The members of the team were Marvin Minsky, Gerhard Rayna, Ariel Zemach; to each of these a prize of thirty dollars is awarded.

The third prize, two hundred dollars, is awarded to the Department of Mathematics of New York University, New York, New York. The members of the team were Sol Ciolkowski, Donald Newman, Elaine Weiss; to each of these a prize of twenty dollars is awarded.

The fourth prize, one hundred dollars, is awarded to the Department of Mathematics of the University of Toronto, Toronto, Ontario, Canada. The members of the team were J. P. Mayberry, R. J. Semple, R. O. Skinner; to each of these a prize of ten dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, were J. P. Mayberry, University of Toronto; Z. Alexander Melzak, McGill University; J. W. Milnor, Princeton University; Donald Newman, New York University; R. J. Semple, University of Toronto. Each of these will receive a prize of fifty dollars.

The five succeeding persons ranking highest in the examination, named in alphabetical order, were R. E. Cutkosky, Carnegie Institute of Technology; D. B. Gillies, University of Toronto; R. S. Pierce, California Institute of Technology; André Robert, Laval University; Ariel Zemach, Harvard College. To each of these a prize of twenty dollars is awarded.

The following teams, named in alphabetical order, won honorable mention: Carnegie Institute of Technology, Pittsburgh, Pennsylvania, the members of the team being R. E. Cutkosky, John S. Nodvik, Douglas H. Shaffer; City College of New York, New York, New York, the members of the team being Arnold Benson, Herman Hanisch, Harold Widom; Cornell University, Ithaca, New York, the members of the team being Carl S. Herz, Michael Horowitz, Walter Meyer; McGill University, Montreal, Quebec, Canada, the members of

* Director of the Competition.

the team being Arthur K. Kerman, Earl L. Lomon, Z. Alexander Melzak.

Eight individuals were given honorable mention. The names are listed in alphabetical order: H. A. Forrester, California Institute of Technology; George Gioumousis, Polytechnic Institute of Brooklyn; Michael Horowitz, Cornell University; R. Sherman Lehman, Stanford University; Paul Martin, Harvard College; Gerhard Rayna, Harvard College; William F. Reynolds, College of the Holy Cross, S. A. Zwick, California Institute of Technology.

The following is a list of all colleges and universities which entered teams in the Competition: Amherst College, Brooklyn College, California Institute of Technology, Carleton College, Carnegie Institute of Technology, City College of New York, College of the Holy Cross, College of St. Thomas, Columbia College, Cornell University, Fordham College, Harvard College, Iowa State College, Laval University, Massachusetts Institute of Technology, McGill University, McMaster University, Montana State College, New York University, Occidental College, Polytechnic Institute of Brooklyn, Princeton University, Providence College, Purdue University, Queen's University, Rockford College, St. Francis College, St. John's College of St. John's University, Stanford University, Swarthmore College, University of British Columbia, University of Buffalo, University of California, University of Detroit, University of Minnesota, University of Oregon, University of Puerto Rico, University of Rochester, University of Toronto, Ursinus College, Yale University.

The following additional colleges and universities entered individual contestants only: Antioch College, College of St. Francis, Incarnate Word College, Saint Joseph College, State Teachers College (Mankato, Minnesota), United States Naval Academy, University of Arkansas, University of California, University of Chicago, University of Oklahoma, University of Omaha, University of South Carolina, University of Wyoming, West Virginia Wesleyan College, Xavier University.

A total of 223 undergraduates representing 56 institutions took part in the Competition.

Participants in the Competition were given the following lists of problems.

Part I. Three Hours

(Answer the questions in any order and by any method. Show all of your work in logical sequence and indicate your answers clearly. No tables or other books may be used. Use the right hand pages of your examination booklet for your solutions, use the left hand pages for scratch work. Cross out any work which you do not wish to have considered. Partial credit may be given on a question, even when the solution is not completed.)

1. For what values of the ratio a/b is the limaçon $r=a-b\cos\theta$ a convex curve? $(a>b>0)$
2. Answer both (a) and (b). Test for convergence the series

(a) $$\frac{1}{\log(2!)}+\frac{1}{\log(3!)}+\frac{1}{\log(4!)}+\cdots+\frac{1}{\log(n!)}+\cdots$$

(b) $\dfrac{1}{3}+\dfrac{1}{3\sqrt{3}}+\dfrac{1}{3\sqrt{3}\sqrt[3]{3}}+\cdots+\dfrac{1}{3\sqrt{3}\sqrt[3]{3}\cdots\sqrt[n]{3}}+\cdots$

3. The sequence $x_0, x_1, x_2, \cdots$ is defined by the conditions

$$x_0 = a, \qquad x_1 = b, \qquad x_{n+1} = \frac{x_{n-1} + (2n-1)x_n}{2n} \qquad \text{for } n \geqq 1,$$

where a and b are given numbers. Express $\lim_{n\to\infty} x_n$ concisely in terms of a and b.

4. Answer either (a) or (b).
   (a) In a right prism with triangular base, given the sum of the areas of three mutually adjacent faces (that is, of two lateral faces and one base), show that these faces are of equal area and perpendicular to each other when the volume attains its maximum.
   (b) Show that

$$\frac{\dfrac{x}{1}+\dfrac{x^3}{1\cdot 3}+\dfrac{x^5}{1\cdot 3\cdot 5}+\dfrac{x^7}{1\cdot 3\cdot 5\cdot 7}+\cdots}{1+\dfrac{x^2}{2}+\dfrac{x^4}{2\cdot 4}+\dfrac{x^6}{2\cdot 4\cdot 6}+\cdots} = \int_0^x e^{-t^2/2}dt.$$

5. A function $D(n)$ of the positive integral variable n is defined by the following properties: $D(1)=0$, $D(p)=1$ if p is a prime, $D(uv)=uD(v)+vD(u)$ for any two positive integers u and v. Answer all three parts below.
   (a) Show that these properties are compatible and determine uniquely $D(n)$. (Derive a formula for $D(n)/n$, assuming that $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ where $p_1, p_2, \cdots, p_k$ are different primes.)
   (b) For what values of n is $D(n)=n$?
   (c) Define $D^2(n)=D[D(n)]$, etc. and find the limit of $D^m(63)$ as m tends to ∞.

6. Each coefficient a_n of the power series

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = f(x)$$

has either the value 1 or the value 0. Prove the easier of the two assertions:
   (a) If $f(0.5)$ is a rational number, $f(x)$ is a rational function.
   (b) If $f(0.5)$ is not a rational number, $f(x)$ is not a rational function.

Part II. Three Hours

(Answer the questions in any order and by any method. Show all of your work in logical sequence and indicate your answers clearly. No tables or other books may be used. Use the right hand pages of your examination booklet for your solutions, use the left hand pages for scratch work. Cross out any work which you do not wish to have considered. Partial credit may be given on a question, even when the solution is not completed.)

1. In each of n houses on a straight street are one or more boys. At what point

should all the boys meet so that the sum of the distances that they walk is as small as possible?

2. Two obvious approximations to the length of the perimeter of the ellipse with semi-axes a and b are $\pi(a+b)$ and $2\pi(ab)^{1/2}$. Which one comes nearer the truth when the ratio b/a is very close to 1?
3. In the Gregorian calendar:
   (a) years not divisible by 4 are common years;
   (b) years divisible by 4 but not by 100 are leap years;
   (c) years divisible by 100 but not by 400 are common years;
   (d) years divisible by 400 are leap years;
   (e) a leap year contains 366 days; a common year 365 days. Prove that the probability that Christmas falls on a Wednesday is not 1/7.
4. The cross-section of a right cylinder is an ellipse, with semi-axes a and b, where $a>b$. The cylinder is very long, made of very light homogeneous material. The cylinder rests on the horizontal ground which it touches along the straight line joining the lower endpoints of the minor axes of its several cross-sections. Along the upper endpoints of these minor axes lies a very heavy homogeneous wire, straight and just as long as the cylinder. The wire and the cylinder are rigidly connected. We neglect the weight of the cylinder, the breadth of the wire and the friction of the ground.

   The system described is in equilibrium, because of its symmetry. This equilibrium seems to be stable when the ratio b/a is very small, but unstable when this ratio comes close to 1. Examine this assertion and find the value of the ratio b/a which separates the cases of stable and unstable equilibrium.
5. Answer either (a) or (b).
   (a) Given that the sequence whose nth term is (s_n+2s_{n+1}) converges, show that the sequence $\{s_n\}$ converges also.
   (b) A plane varies so that it includes a cone of constant volume equal to $\pi a^3/3$ with the surface the equation of which in rectangular coordinates is $2xy=z^2$. Find the equation of the envelope of the various positions of this plane.

   State the result so that it applies to a general cone (that is, conic surface) of the second order.
6. Consider the closed plane curves C_i and C_o, their respective lengths $|C_i|$ and $|C_o|$, the closed surfaces S_i and S_o, and their respective areas $|S_i|$ and $|S_o|$. Assume that C_i lies inside C_o and S_i inside S_o. (Subscript i stands for "inner," o for "outer.") Prove the correct assertions among the following four, and disprove the others.
   (a) If C_i is convex, $|C_i| \leqq |C_o|$.
   (b) If S_i is convex, $|S_i| \leqq |S_o|$.
   (c) If C_o is the smallest convex curve containing C_i, then $|C_o| \leqq |C_i|$.
   (d) If S_o is the smallest convex surface containing S_i, then $|S_o| \leqq |S_i|$.

   You may assume that C_i and C_o are polygons and S_i and S_o polyhedra. (Why?)

MATHEMATICAL NOTES

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ON THE GREEN FUNCTION OF A STAR-SHAPED THREE DIMENSIONAL REGION

S. E. WARSCHAWSKI, University of Minnesota

1. Introduction. In 1931 J. J. Gergen[1] proved the following theorem.

THEOREM. *Suppose the region D in the (x, y, z)-space is star-shaped with regard to the origin O, and that $g(x, y, z)$ is the Green function of D with pole at O. Then at every point in $D-O$ the square of the gradient $(\nabla g)^2 = g_x^2 + g_y^2 + g_z^2$ satisfies the condition*

$$(1) \qquad (\nabla g)^2 \geqq \frac{g^2}{r^2} > 0, \qquad (r = \sqrt{x^2 + y^2 + z^2}),$$

the inequality holding throughout $D-O$, unless $g - 1/r \equiv 0$. Furthermore, for every positive constant c, the region determined by the inequality $g > c$ is star-shaped with respect to O.

The function $g(P) = g(x, y, z)$ is the Green function of first kind of Laplace's equation, if the latter exists. If this is not the case, then $g(P)$ denotes the "*extended Green function*," which is defined as follows:[2] Let D_n $(n = 1, 2, \cdots)$ be a sequence of regions such that: (i) $O \in D_n \subset D_{n+1} \subset D$, (ii) the D_n exhaust the region D, (iii) the Green function $g_n(P)$, with pole at O, exists for each D. Then for every $P \in D$, $\lim_{n\to\infty} g_n(P) = g(P)$ exists, uniformly in any closed bounded subset of D; $g(P)$ is the extended Green function of D. This function always exists, is independent of the choice of the sequence D_n, and it coincides with the Green function in the ordinary sense, if the latter exists.

In the following we give a short, elementary proof of Gergen's theorem.

2. Proof. Let λ be a *constant*, $0 < \lambda < 1$. The function

$$h(P) = \frac{g(P) - \lambda g(\lambda P)}{1 - \lambda} \qquad (\lambda P \text{ is the point } (\lambda x, \lambda y, \lambda z))$$

is harmonic for $P \in D$, including the point O, if properly defined at O, since

$$(1 - \lambda)h(P) = \frac{1}{r} + v(P) - \left[\frac{1}{r} + \lambda v(\lambda P)\right] = v(P) - \lambda v(\lambda P),$$

[1] Note on the Green function of a star-shaped three dimensional region, American Journal of Mathematics, vol. 42 (1931), pp. 746–752.

[2] See, for example, O. D. Kellogg, Recent progress with the Dirichlet Problem, Bulletin of the American Mathematical Society, vol. 32 (1926), pp. 601–625. See in particular p. 604.

where $v(P)$ is harmonic in D.

We assume at first that the Green function of D exists in the proper sense. Then $\lim g(P)=0$ as P approaches any boundary point P_0 of D. Hence

$$\lim_{P\to P_0} h(P) = -\frac{\lambda g(\lambda P_0)}{1-\lambda} \leqq 0.$$

By the principle of the maximum we have, therefore, for any $P\in D$,

$$\frac{g(P)-\lambda g(\lambda P)}{1-\lambda} \leqq 0. \tag{2}$$

Both statements of the theorem follow at once from (2). If $g(P)>c>0$, then by (2) we have

$$g(\lambda P) \geqq \frac{g(P)}{\lambda} > \frac{c}{\lambda} > c;$$

that is, with P, the point λP, $0<\lambda<1$, is also in the region $g>c$; the region $g>c$ is star-shaped with respect to O.

To prove (1) we let, for *fixed* $P\in D$, $(P\neq O)$, $\lambda\to 1$ and find, from (2),

$$\lim_{\lambda\to 1} \frac{rg(P)-r\lambda g(\lambda P)}{r(1-\lambda)} = \frac{\partial(rg)}{\partial r}\bigg|_P \leqq 0,$$

or

$$g + r\frac{\partial g}{\partial r} \leqq 0,$$

whence

$$\frac{g}{r} \leqq -\frac{\partial g}{\partial r} = \left|\frac{\partial g}{\partial r}\right| \leqq |\nabla g|,$$

since

$$\left|\frac{\partial g}{\partial r}\right| = \left|g_x\frac{x}{r} + g_y\frac{y}{r} + g_z\frac{z}{r}\right| \leqq |\nabla g|.$$

If the equality in (1) holds for just one point $P_0\in D$, then $g\equiv 1/r$. For since $\partial(rg)/\partial r$ is $\leqq 0$ and harmonic in D, it must be identically equal to 0, if it vanishes at an interior point P_0 of D. Thus[3]

$$g + r\frac{\partial g}{\partial r} = v + xv_x + yv_y + zv_z \equiv 0, \qquad P\in D.$$

Hence $v(O)=0$. But since $v=g-1/r$ is non-positive in D (this is also true when

[3] This is merely a repetition of Gergen's argument, *loc. cit.*,[1] p. 751 (bottom of page).

g is the *extended* function of Green), the fact that $v(O)=0$ implies $v(P)\equiv 0$, $g(P)\equiv 1/r$.

3. The general case. In order to prove the theorem in the case that only the extended function of Green exists, it is sufficient to show that (2) remains valid when $g(P)$ is the latter function. Assume at present that D is *bounded*. For a *fixed* λ, $0<\lambda<1$, let $D(\lambda)$ denote the region consisting of all points $\lambda P(\lambda x, \lambda y, \lambda z)$, where $P(x, y, z)$ is in D. Since D is star-shaped and *bounded*, the *closure* $\overline{D}(\lambda)$ of $D(\lambda)$ is contained in D. Let D_n be a sequence of regions lying in D used in the definition of the extended Green function $g(P)$ of D. For sufficiently large n, $D_n \supset \overline{D}(\lambda)$, so that, in particular, for $P\in D_n$, we have $\lambda P\in D_n$. Let $g_n(P)$ denote the (proper) Green function of D_n. Then

$$h_n(P) = \frac{g_n(P) - \lambda g_n(\lambda P)}{1-\lambda}$$

is harmonic in D_n, and it follows just as above that $h_n(P)\leqq 0$ for every $P\in D_n$. Now we keep P *fixed* and let $n\to\infty$; since $\lim_{n\to\infty} g_n(P)=g(P)$, we obtain (2), valid for every $P\in D$ and every λ, $0<\lambda<1$.

If D is *not bounded*, let Δ_n be the intersection of D with the sphere $x^2+y^2+z^2 < R_n^2$, $n=1, 2, \cdots$, where $R_n\to\infty$ as $n\to\infty$. Denote by $\gamma_n(P)$ the Green function (extended or proper, if it exists) of Δ_n with pole at O. Then it is easily seen[4] that, for $P\in D$, $P\neq O$, we have

$$\lim_{n\to\infty} \gamma_n(P) = g(P),$$

where $g(P)$ is the function of Green of the *unbounded* region D. Since each Δ_n is star-shaped with regard to O and bounded, (2) holds, with γ_n in place of g, for $P\in\Delta_n$, and therefore, as $n\to\infty$, we see that (2) is true also for the case that D is not bounded.

ON ANALYTIC MAPS OF CIRCLES INTO CONVEX REGIONS

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1. Theorem. In this note we shall establish the following result.

THEOREM. *Let a function $w=f(z)$ have the properties that it is analytic in the open unit circle and that the map of the unit circle is contained in a convex region R. Then the map of the circle $|z|\leqq r(r<1)$ is contained in a convex region obtained by deforming R homothetically, with respect to the map of the origin, in the ratio $2r/(r+1)$.*

[4] Let D_n^* be a sequence of regions which is used in the definition of the Green function $g(P)$ of D. We may assume that $D_n^*\subset\Delta_n$ $(n=1, 2, \cdots)$. If $g_n^*(P)$ is the (proper) Green function of D_n^* with pole at O, then $\lim_{n\to\infty} g_n^*(P)=g(P)$. Furthermore, since $D_n^*\subset\Delta_n\subset D$, we have

$$g_n^*(P) \leqq \gamma_n(P) \leqq g(P),$$

so that $\lim_{n\to\infty}\gamma_n(P)=g(P)$.

2. Proof. Without loss of generality we may assume that $f(0)=0$. Let $w=u+iv$. Then the function $h_\phi(x, y)=u\cos\phi+v\sin\phi$ has a maximum $h(\phi)$, the support function for the convex hull H of the map $f(U)$ of the open unit circle U. If the convex hull H contains the half ray from the origin in the direction ϕ_0, then $h(\phi_0)$ is infinite. For each fixed ϕ, we have that $h_\phi(x, y)$ is harmonic and that $h_\phi(0, 0)=0$. We consider the function $g(z)=2z/(z+1)$, which maps the open unit circle onto the half plane $\Re(w)<1$ and maps zero onto zero. Since the maximum of the real part of $g(z)$ on $|z|=r$ is $2r/(r+1)$, it follows that for every value ϕ the inequality $|z|\leqq r$ implies that $h_\phi(x, y)\leqq[2r/(r+1)]h(\phi)$. If we define $S(\phi)$ as the support function for the region R and $h_r(\phi)$ as the support function for the map of the circle $|z|\leqq r$, then we have $h_r(\phi)\leqq[2r/(r+1)]S(\phi)$. For more details concerning the ideas of the proof, see Pólya and Szegö.*

3. The extreme case. That $2r/(r+1)$ is the best possible ratio may be seen directly by taking $f(z)=2z/(z+1)$.

In addition, for the functions $w=(z-\alpha)/(\bar{\alpha}z-1)$, $(|\alpha|<1)$, mapping the unit circle onto the unit circle, the hypotheses of the theorem apply if the region R is taken as the open unit circle. The best deformation ratio for each of these functions is $(|\alpha|+1)r/(|\alpha|r+1)$. Since the least upper bound of these ratios for $|\alpha|<1$ is $2r/(r+1)$, we have obtained the best possible ratio, even if we are restricted to *bounded* convex regions.

If the region R is symmetric with respect to the map of the origin, then the deformation ratio can be improved to $(4/\pi)$ (arc tan r).*

CONFORMAL MAPPING OF THE INTERIOR OF AN ELLIPSE ONTO A CIRCLE

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This mapping can be found in the paper of H. A. Schwarz, *Über einige Abbildungsaufgaben*, Journal für die reine und angew. Mathematik, vol. 70 (1869), pp. 105–120; *Gesammelte Abhandlungen*, vol. 2, pp. 65–83. (See, in particular, pp. 77–78.) The mapping function given by Schwarz appears in terms of a Jacobian elliptic function. Our purpose is to deal with this problem in an elementary way. Later we shall compare our result with that of Schwarz.

1. We assume that the given ellipse E is in the complex z-plane, with foci at ±1 and with semiaxes a and b, $a>b$, $a^2-b^2=1$. We introduce a second complex variable w defined by the elementary mapping

$$2z = w + w^{-1}. \tag{1}$$

It is well known that the circle $|w|=R=a+b$, $R>1$, is transformed into E. We denote by

$$z' = f(z) = c_1z + c_2z^2 + \cdots, \qquad c_1 > 0, \tag{2}$$

* Pólya and Szegö, Aufgaben und Lehrsatze aus der Analysis, Berlin, Springer, 1925, vol. 1, pp. 106, 139, and 140.

the mapping function in question, so that $f(z)$ is schlicht in the interior of E and $|f(z)| = 1$ on E. Hence

$$F(z) = \log [f(z)/z] \tag{3}$$

is regular in the interior of E (real at $z=0$). The function $F(z)$ can be expanded in a series of Tchebychev polynomials $T_n(z)$ defined by

$$2T_n(z) = w^n + w^{-n}. \tag{4}$$

For this theorem, see for instance Picard, *Traité d'Analyse*, 3rd edition, vol. 2, p. 336. It follows immediately from the fact that the function $F(\frac{1}{2}(w+w^{-1}))$ is single-valued and analytic in the circular ring

$$R^{-1} < |w| < R$$

and remains unchanged if we replace w by w^{-1}. Hence it can be expanded in a Laurent series

$$F(\tfrac{1}{2}(w + w^{-1})) = \sum_{n=-\infty}^{\infty} d_n w^n,$$

in which $d_{-n} = d_n$.

We write:

$$\log \frac{f(z)}{z} = \sum_{n=0}^{\infty} A_n T_n(z). \tag{5}$$

Obviously A_n is real.

2. Let $|w| = R$, $w = Re^{i\phi}$. Taking real parts on both sides of (5) we have, since $|f(z)| = 1$,

$$\log \frac{1}{|z|} = \log \frac{2}{|Re^{i\phi} + R^{-1}e^{-i\phi}|} = \sum_{n=0}^{\infty} A_n \frac{R^n + R^{-n}}{2} \cos n\phi.$$

Now

$$\begin{aligned} \log \frac{2}{|Re^{i\phi} + R^{-1}e^{-i\phi}|} &= \log \frac{2}{R} + \log \frac{1}{|1 + R^{-2}e^{-2i\phi}|} \\ &= \log \frac{2}{R} + \Re \log \frac{1}{1 + R^{-2}e^{-2i\phi}} \\ &= \log \frac{2}{R} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} R^{-2n} \cos 2n\phi. \end{aligned}$$

Comparing this with (5) we find

$$\begin{aligned} &A_0 = \log \frac{2}{R}, \qquad A_{2n} \frac{R^{2n} + R^{-2n}}{2} = \frac{(-1)^n}{n} R^{-2n}, \\ &A_{2n} = \frac{(-1)^n}{n} \frac{2R^{-2n}}{R^{2n} + R^{-2n}}, \qquad A_{2n-1} = 0; \qquad n = 1, 2, 3, \cdots. \end{aligned} \tag{6}$$

Hence the following representation of the mapping function arises:

$$\log \frac{f(z)}{z} = \log \frac{2}{R} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{2R^{-2n}}{R^{2n} + R^{-2n}} T_{2n}(z). \tag{7}$$

The first coefficient c_1 of $f(z)$ is r_0^{-1}, where r_0 is the "inner radius" of the ellipse with respect to the origin. In view of the relation

$$T_{2n}(0) = (-1)^n,$$

we obtain

$$\log r_0 = \log \frac{R}{2} - \sum_{n=1}^{\infty} \frac{1}{n} \frac{2R^{-2n}}{R^{2n} + R^{-2n}}. \tag{8}$$

Incidentally, the "outer radius" of the ellipse is

$$\bar{r} = \tfrac{1}{2}(a + b) = \frac{R}{2},$$

and from (8) we see that $r_0 < \bar{r}$.

3. The function

$$G = \Re \log \frac{1}{f(z)} = \log \frac{1}{|f(z)|} \tag{9}$$

is the classical Green function of the interior of E with respect to the origin. We can prove in the customary way (forming the conjugate of G) that the mapping $z' = f(z)$ carries over the interior of E in a schlicht manner onto $|z'| < 1$. Another, direct procedure would be the following.

We want to show that if z describes the boundary of E, z' describes the unit circle $|z'| = 1$ exactly once and in the same direction. (Then we apply the argument principle.) For this purpose we form the imaginary part of (7):

$$\begin{aligned} \Im \log f(z) &= \Im \log z + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{2R^{-2n}}{R^{2n} + R^{-2n}} \Im T_{2n}(z) \\ &= \Im \log \frac{Re^{i\phi} + R^{-1}e^{-i\phi}}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{2R^{-2n}}{R^{2n} + R^{-2n}} \frac{R^{2n} - R^{-2n}}{2} \sin 2n\phi, \\ \frac{d}{d\phi} \Im \log f(z) &= \Im \frac{i(Re^{i\phi} - R^{-1}e^{-i\phi})}{Re^{i\phi} + R^{-1}e^{-i\phi}} + 2\sum_{n=1}^{\infty} (-1)^n \frac{R^{2n} - R^{-2n}}{R^{2n} + R^{-2n}} R^{-2n} \cos 2n\phi. \end{aligned} \tag{10}$$

The first term on the right-hand side can be written as follows:

$$\Re \left(\frac{1 - R^{-2}e^{-2i\phi}}{1 + R^{-2}e^{-2i\phi}} \right) = 1 + 2\sum_{n=1}^{\infty} (-1)^n R^{-2n} \cos 2n\phi, \tag{11}$$

so that

$$\begin{aligned} \frac{d}{d\phi} \Im \log f(z) &= 1 + 2\sum_{n=1}^{\infty} (-1)^n \frac{2}{R^{2n} + R^{-2n}} \cos 2n\phi \\ &= \lim_{\rho \to 1-0} \left(1 + 2\sum_{n=1}^{\infty} (-1)^n \frac{2\rho^n}{R^{2n} + R^{-2n}} \cos 2n\phi \right). \end{aligned} \tag{12}$$

We define α by $e^{\alpha\pi} = R^2$ and use the formula

$$\frac{\pi}{\operatorname{ch} \alpha\pi} = \int_0^\infty \frac{\cos \alpha x}{\operatorname{ch} (x/2)}\, dx, \tag{13}$$

so that

$$\frac{2}{R^{2n} + R^{-2n}} = \frac{1}{\operatorname{ch} (n\alpha\pi)} = \frac{1}{\pi} \int_0^\infty \frac{\cos n\alpha x}{\operatorname{ch} (x/2)}\, dx, \qquad n = 0, 1, 2, \cdots. \tag{14}$$

Hence

$$\begin{aligned} 1 + 2\sum_{n=1}^{\infty} (-1)^n \frac{2\rho^n}{R^{2n} + R^{-2n}} \cos 2n\phi &= \frac{1}{\pi} \int_0^\infty \frac{dx}{\operatorname{ch} (x/2)} \left(1 + 2\sum_{n=1}^{\infty} (-\rho)^n \cos n\alpha x \cos 2n\phi\right) \\ &= \frac{1}{2\pi} \int_0^\infty \frac{dx}{\operatorname{ch} (x/2)} \left(\frac{1 - \rho^2}{1 + 2\rho \cos (\alpha x + 2\phi) + \rho^2} + \frac{1 - \rho^2}{1 + 2\rho \cos (\alpha x - 2\phi) + \rho^2}\right) > 0. \end{aligned} \tag{15}$$

4. Finally we compare our main formula (7) with the formula of Schwarz given in the paper quoted above. As to the elliptic and theta functions occurring below, we use the notation of Whittaker & Watson, *Modern Analysis*, American edition, 1943.

Introducing the notation

$$w = e^{i(\pi/2 - z)} = ie^{-ix}, \qquad z = \cos (\pi/2 - x) = \sin x, \tag{16}$$

we find

$$\log \frac{f(z)}{z} = \log \frac{2}{R} + \sum_{n=1}^{\infty} \frac{1}{n} \frac{2R^{-2n}}{R^{2n} + R^{-2n}} \cos 2nx. \tag{17}$$

But according to a formula of Jacobi (Fundamenta Nova, p. 99; Whittaker & Watson, p. 509, Example 3):

$$\sum_{n=1}^{\infty} \frac{1}{n} \frac{2R^{-4n}}{1 + R^{-4n}} \cos 2nx = \log \operatorname{sn} \frac{2Kx}{\pi} - \log (2q^{1/4}) + \tfrac{1}{2} \log k - \log \sin x \tag{18}$$

where

$$R^{-4} = q. \tag{19}$$

Consequently

$$\log \frac{f(z)}{\sin x} = \log \frac{2}{R} - \log (2q^{1/4}) + \tfrac{1}{2} \log k - \log \sin x + \log \operatorname{sn} \frac{2Kx}{\pi}, \tag{20}$$

$$\boxed{f(z) = k^{1/2} \operatorname{sn}\left(\frac{2K}{\pi} \sin^{-1} z\right).}$$

This is the formula of Schwarz. For the sake of completeness we quote the well-known expressions for the modulus k in terms of q:

$$k = \left(\frac{\theta_2}{\theta_3}\right)^2 = \left(\frac{2q^{1/4} + 2q^{9/4} + 2q^{25/4} + \cdots}{1 + 2q + 2q^4 + 2q^9 + \cdots}\right)^2 \tag{21}$$
$$= 4q^{1/2}\left(\frac{(1+q^2)(1+q^4)(1+q^6)\cdots}{(1+q)(1+q^3)(1+q^5)\cdots}\right)^4.$$

Also

$$K = \tfrac{1}{2}\pi\theta_3^2 = \tfrac{1}{2}\pi(1 + 2q + 2q^4 + 2q^9 + \cdots)^2 \tag{22}$$
$$= \tfrac{1}{2}\pi((1-q^2)(1-q^4)(1-q^6)\cdots)^2((1+q)(1+q^3)(1+q^5)\cdots)^4,$$
$$r_0 = \left(k^{1/2}\frac{2K}{\pi}\right)^{-1}(\theta_2\theta_3)^{-1}$$
$$= (2q^{1/4} + 2q^{9/4} + 2q^{25/4} + \cdots)^{-1}(1 + 2q + 2q^4 + 2q^9 + \cdots)^{-1}$$
$$= \frac{1 - 2q + 2q^4 - 2q^9 + \cdots}{2q^{1/4} - 3\cdot 2q^{9/4} + 5\cdot 2q^{25/4} - \cdots}$$
$$= \frac{1}{2q^{1/4}}\left(\frac{(1-q)(1-q^3)(1-q^5)\cdots}{(1-q^2)(1-q^4)(a-q^6)\cdots}\right)^2.$$

A NOTE ON THE POISSON KERNEL

A. E. Taylor, University of California, Los Angeles

1. Introduction. The Poisson kernel in two dimensions is defined by

$$P(r, \theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}. \tag{1}$$

The purpose of this note is to give an interesting, and perhaps new, proof of the formula

$$\frac{1}{2\pi}\int_0^{2\pi} P(r, \theta)d\theta = 1, \qquad 0 \leqq r < 1. \tag{2}$$

2. Proof. We observe first of all that if we denote the integral on the left in (2) by $F(r)$, the function of r so defined is continuous when $0 \leqq r < 1$. Also, it is immediately clear that $F(0) = 1$.

Now let us write

$$F(r) = \frac{1}{2\pi}\int_0^{\pi} P(r, \theta)d\theta + \frac{1}{2\pi}\int_{\pi}^{2\pi} P(r, x)dx.$$

Making the substitution $x = \theta + \pi$ in the second integral, and then combining the two integrals, we find

$$F(r) = \frac{1}{\pi}\int_0^{\pi} \frac{1 - r^4}{(1 + r^2)^2 - 4r^2\cos^2\theta}\,d\theta.$$

Next we observe that

$$(1 + r^2)^2 - 4r^2 \cos^2\theta = 1 - 2r^2(2\cos^2\theta - 1) + r^4,$$

and that $2\cos^2\theta - 1 = \cos 2\theta$. Hence, setting $\phi = 2\theta$, we find

$$F(r) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1 - r^4}{1 - 2r^2\cos\phi + r^4}\, d\phi = F(r^2).$$

By repeated applications of this last result we see that $F(r) = F(r^{2^n})$, $n = 1, 2, \cdots$. Therefore, if $0 \leqq r < 1$, we have

$$F(r) = \lim_{n\to\infty} F(r^{2^n}) = F(0) = 1;$$

thus (2) is proved.

Finally, we observe that $F(r) = -F(1/r)$ if $0 < r \neq 1$. From this result and (2) we infer the formula

$$\frac{1}{2\pi}\int_0^{2\pi} P(r, \theta)\, d\theta = -1, \qquad 1 < r.$$

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

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ON INTEGRATION BY PARTS

T. E. SYDNOR, Pasadena City College

If $\int dx/x \log x$ is integrated by parts using $u = 1/\log x$ and $dv = dx/x$ in $\int u\, dv = uv - \int v\, du + c$ the well-known but maddening identity $0 = 0$ results.

More generally the method of integration by parts of $\int f_1(x) f_2(x)\, dx$ becomes useless if $f_1(x) \int f_2(x)\, dx = K$ (constant). Such integrals, however, can easily be integrated, for this condition implies that:

$$f_2(x) = -Kf_1'(x)/(f_1(x))^2.$$

So

$$\int f_1(x)f_2(x)dx = -K\int f_1'(x)dx/f_1(x) = -K\log f_1(x) + C.$$

SYNTHETIC TECHNIQUE FOR PARTIAL FRACTIONS

J. P. BALLANTINE, University of Washington

In the March 1950 issue of this MONTHLY, M. R. Spiegel showed by a clever device how to reduce the tedium of resolving a particular rational function into partial fractions. I also deviate from the standard procedure. By my method, the computation for the same function appears as follows:

$$\frac{3x+1}{(x-2)(x^2+x+1)^3} = \frac{A}{x-2} + R(x),$$

where R is a rational function of x whose denominator does not contain the factor $x-2$, and A is seen to be $1/49$.

First solve the equation for $R(x)$:

$$R(x) = \frac{49(3x+1) - (x^2+x+1)^3}{49(x-2)(x^2+x+1)^3} = \frac{N}{49 \cdot B \cdot T^3}.$$

The notations N, B, and T, short for numerator, binomial and trinomial, are merely to explain the following synthetic computation.

$T^2 =$			1	+2	+3	+2	+1	
$x \cdot T^2 =$		1	+2	+3	+2	+1	0	
$x^2 \cdot T^2 =$	1	+2	+3	+2	+1	0	0	
$T^3 =$	1	+3	+6	+7	+6	+3	+ 1	
$49(3x+1) =$						147	+49	
$N =$	−1	−3	−6	−7	−6	+144	+48(2	
		−2	−10	−32	−78	−168	−48	
$N/B =$	−1	−5	−16	−39	−84	−24	(−1	−1
		+1	+4	+11	+24			
			+1	+4	+11	+24		
	−1	−4	−11	−24	/ −49	0		

The last step was a quadratic synthetic division, by x^2+x+1. When dividing by $x-r$, the divisor is written as merely r, so when dividing by x^2-rx-s, the divisor is written as merely $+r\ +s$.

The last two numbers, -49 and 0, are the coefficients of the remainder, namely $-49x+0$. The quotient is $-x^3-4x^2-11x-24$. The process continues, without break:

$$\begin{array}{cccccc} -1 & -4 & -11 & -24 & (-1 & -1 \\ & 1 & +3 & & & \\ & & 1 & +3 & & \\ \hline -1 & -3\ / & -7 & -21 & & \end{array}$$

$$R(x) = \frac{-x-3}{49T} + \frac{-7x-21}{49T^2} + \frac{-49x}{49T^3}.$$

This is the final answer, except for the term $1/49\,(x-2)$.

ON SPACE FRAMES AND PLANE TRUSSES

H. E. GOHEEN, Syracuse University

The theory of statically determinate trusses and space frames as usually given in textbooks on mechanics* is difficult for students beyond the necessary difficulty of any mathematical discipline. An understanding of the general theory of systems of n linear algebraic equations in m unknowns, and the proper definition of "statically determinate," will, in the opinion of the author, lead to a better understanding of this subject matter. The following discussion is believed an improvement.

For definitions of a statically determinate space frame and statically determinate plane trusses, we use the following:

A space frame is statically determinate if and only if the compression in all members and the reaction forces necessary to support the frame in equilibrium as a rigid body in space are uniquely determined for any arbitrary space loading from the set of space equilibrium conditions, three at each joint of the frame.

A plane truss is statically determinate if and only if the compressions in all members and the reaction forces necessary to support the truss in equilibrium as a rigid body in the plane are uniquely determined for any arbitrary plane loading from the set of plane equilibrium conditions, two at each joint of the truss.

In space the condition that the resultant force is zero at the kth joint yields three linear equations

$$\text{I} \qquad \begin{aligned} \sum \alpha_{ik} F_i &= -X_k, \\ \sum \beta_{ik} F_i &= -Y_k, \\ \sum \gamma_{ik} F_i &= -Z_k, \end{aligned}$$

in which α_{ik}, β_{ik}, and γ_{ik} are the direction cosines of the ith force (either reaction force or bar compression force) at the kth joint if the ith force is concurrent with that joint and are otherwise taken 0, while X_k, Y_k, and Z_k are the X, Y, and Z

* See for example Timoshenko and Young, Statics, first edition, New York, McGraw-Hill (1937), Chapters II, V, and VII.

components of the loading force at the kth joint.

In the plane truss the equilibrium conditions at each joint are two in number. They are

$$\text{II} \qquad \begin{aligned} \sum \alpha_{ik} F_k &= -Y_k \\ \sum \beta_{ik} F_k &= -X_k \end{aligned}$$

in which α_{ik} and β_{ik} are the sine and cosine respectively of the angle which the ith force makes with the X-axis at the kth joint, if it is concurrent with the kth joint and zero otherwise, while X_k and Y_k are the X and Y components of the loading at the kth joint.

To support a rigid body in space three restraints are necessary to locate a point, two more to locate a line, and one more to prevent rotation about that line. Against each restraint, there will be a reaction force if the body is to be in equilibrium. In each member of the frame the bar force at one end is equal and opposite the force at the other end. Hence there are m unknown bar forces and six unknown reaction forces. We assume that these are j joints.

Thus the system (I) is a system of $3j$ equations in $m+6$ unknowns. If we note that the right hand sides of (I) are arbitrary and appeal to the theorem that the system $\sum_{k=1}^{n} a_{ik}x_k = c_i$ $(i = 1, 2, \cdots, m)$ has unique solution for $x_1, x_2, \cdots, x_n$ for arbitrary c_i if and only if $m = n$ and the linear forms $\sum_{k=1}^{n} a_{ik}x_k$ are linearly independent, we have the result, proved in most textbooks by considering only simple frames, that a necessary condition for a space frame to be statically determinate is that

$$m = 3j - 6.$$

Likewise since two restraints are necessary to locate a point in a plane and one more to prevent rotation about that point, there will be three unknown reaction forces at the supports of a plane truss and together with the m unknown bar forces there will be $m+3$ unknowns in the system of $2j$ equations (II). Thus in a plane statically determinate truss we must have

$$m = 2j - 3.$$

If the condition $m = 3j-6$ is fulfilled for a space frame and if relative motion of any two joints of it is impossible, then it is statically determinate. For since relative motion is impossible there must exist solutions of the $3j$ equilibrium equations for arbitrary loading. Since there is a solution for arbitrary loading the equations must be consistent for arbitrary right hand sides and since the number of equations is equal to the number of unknowns there must be unique solution for each loading. Hence, by definition, the frame is statically determinate. Likewise in the plane, a truss is statically determinate if $m = 2j-3$ and relative motion of any two joints is impossible.

Thus in particular any simple frame or simple truss is statically determinate. More complicated structures must be analyzed by more complicated methods.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 926. *Proposed by H. L. Lee, University of Tennessee*

Show that there exist arithmetic progressions with integral terms such that the sum of the squares of four consecutive terms is a 2^kth power of an integer, k a non-negative integer.

E 927. *Proposed by J. H. Braun, Illinois Institute of Technology*

In a regular polygon of $2n+1$ sides all the diagonals are drawn. Find, as a function of n, the total number of triangles of all shapes and sizes thus formed.

E 928. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given a tetrahedron $ABCD$ and a point O. Denote by A', B', C', D' the intersections of AO, BO, CO, DO with the corresponding faces of the tetrahedron, and set $x=AO/A'O$, $y=BO/B'O$, $z=CO/C'O$, $t=DO/D'O$. Show that

$$xyzt = 3 - 2(x + y + z + t) + (xy + xz + xt + yz + yt + zt).$$

E 929. *Proposed by H. S. Shapiro, Massachusetts Institute of Technology*

Given three non-concurrent straight lines l_1, l_2, l_3 in the plane. Let T_i denote reflection in l_i and set $T=T_1T_2T_3$. Show that T^2 is a translation.

E 930. *Proposed by Gordon Raisbeck, Bell Telephone Laboratories*

If

$$\prod_{i=1}^{n} (x + r_i) \equiv \sum_{i=0}^{n} a_i x^{n-i},$$

show that

$$\sum_{i=1}^{n} \tan^{-1} r_i = \tan^{-1} \frac{a_1 - a_3 + a_5 - \cdots}{a_0 - a_2 + a_4 - \cdots}$$

and

$$\sum_{i=1}^{n} \tanh^{-1} r_i = \tanh^{-1} \frac{a_1 + a_3 + a_5 + \cdots}{a_0 + a_2 + a_4 + \cdots}.$$

SOLUTIONS

E 884 [1950, 264]. Correction

The last sign in the quintic given in the note to the solution of this problem should be + instead of −.

Necklace Permutations

E 28 [1933, 175]. *Proposed by H. D. Ruderman, Manhattan High School of Aviation Trades, New York, N. Y.*

Given m marbles of which m_1 are of a first color, m_2 of a second color, m_3 of a third color, and so on to m_n of an n-th color, all arranged in a circle. Determine the number of different arrangements possible if we can start from any point and move in either direction around the circle.

Solution by H. S. Grant, Rutgers University. This is essentially the problem of determining the number of "necklace" permutations, the solution to which appeared in *On a formula for circular permutations*, Mathematics Magazine, Jan.–Feb. 1950, pp. 133–136. If P denote the number of circular permutations, x the number of odd $m_i, j, k, l, \cdots$ the subscripts for the odd m_i, and the modified Kronecker delta $\delta^x_{abc\ldots}$ is defined as 1 when x is any one of $a, b, c, \cdots$, and as 0 otherwise (including the case in which there is no subscript), the formula is

$$P/2 + \delta^x_{012}[(m - \delta^x_1 - 2\delta^x_2)/2]!/2 \prod_{i=1}^{n} [(m_i - \delta^i_{jkl\cdots})/2]!.$$

The formula for P was found in the solution to Problem 519, *National Mathematics Magazine*, April 1944, pp. 276–287, and also in the solution to E 678 [1946, 101]. This formula is given below, D denoting the greatest common divisor of the m_i's, $\phi(d)$ Euler's ϕ-function, and the summation is to be taken over all divisors d of D:

$$P = (1/m) \sum_{d \mid D} \phi(d) \left[(m/d)!/ \prod_{i=1}^{n} (m_i/d)! \right].$$

The reader is also referred to a paper by R. E. Allardice, *On a problem in permutations*, Proceedings of the Edinburgh Mathematical Society, vol. 8 (1890), pp. 64–69. This problem had been suggested by Professor Chrystal.

Also solved by Octave Levenspiel.

For a special case of this problem see E 599 [1944, 472].

Lines Cutting Off One-third the Area of a Triangle

E 774 [1947, 281]. *Proposed by Norman Anning, Ann Arbor, Michigan*

Consider points on the median of a triangle. Through the centroid no straight line can be drawn which will cut off one-third of the area. Through a point four-

fifths of the distance from vertex to base, four such lines can be drawn. Find points on the median at which the number of possible lines changes.

Solution by Vern Hoggatt, Oregon State College. The following solution is largely compounded from the partial results sent to the editor by Free Jamison and C. S. Ogilvy. It is a pleasure to acknowledge the assistance furnished by these results.

Using oblique cartesian coordinates let the vertices A, B, C of the triangle be $(0, b)$, $(a, 0)$, $(0, 0)$. Choose the median through A and consider first a line that intersects the side to which the median is drawn and one of the other sides, say the y-axis. The equation of the chosen median is

$$x = (ab - ay)/2b. \tag{1}$$

The equation of the line passing through $(ar, 0)$ and $(0, b/3r)$, and consequently cutting off one-third of the area, is

$$x = (arb - 3ar^2y)/b. \tag{2}$$

The equation of another line cutting off one-third of the area is

$$x = (2arb - 3ar^2y)/2b. \tag{3}$$

Solving (1) and (2) simultaneously we find $y/b=(1-2r)/(1-6r^2)$, which has a maximum of $(3-\sqrt{3})/6$ when $r=(3+\sqrt{3})/6$. As r varies from $1/2$ to $(3+\sqrt{3})/6$ to 1, y/b varies from 0 to $(3-\sqrt{3})/6$ to $1/5$. Thus, if z is the fractional part of the median from the vertex to the intersection of (1) and (2), we have $z=1-y/b$, and for $z=(3+\sqrt{3})/6$ there is one line, for $(3+\sqrt{3})/6<z\leqq 4/5$ there are two lines, and for $4/5<z\leqq 1$ there is one line. The same results may be obtained by letting the line intersect the third side of the triangle, whence the above number of lines under each condition is to be doubled.

Solving (1) and (3) simultaneously we find $y/b=(1-2r)/(1-3r^2)$, which is a single-valued function of r with no turning-point maximum or minimum between $r=2/3$ and $r=1$. The end-point minimum at $r=1$ gives $y/b=1/2$, and $r=2/3$ gives $y/b=1$. Thus for $0\leqq z\leqq 1/2$ there are two lines.

Now choose another oblique cartesian frame of reference so that the vertices A, B, C are represented by $(0, 0)$, $(c, 0)$, $(0, b)$. The equation of the median through A is now

$$x = cy/b, \tag{4}$$

and we have, for lines analogous to lines (2) and (3),

$$x = (crb - 3cr^2y)/b, \tag{2'}$$

$$x = (2crb - 3cr^2y)/2b. \tag{3'}$$

We now have $z=2y/b$.

Solving (4) and (2′) simultaneously we find $2y/b=2r/(1+3r^2)$, which has a maximum of $\sqrt{3}/3$ at $r=\sqrt{3}/3$, since r must be non-negative. At $r=1$ and

$r=1/3$, $2y/b=1/2$, and consequently at $z=\sqrt{3}/3$ there is one line, and for $1/2 \leqq z \leqq \sqrt{3}/3$ there are two lines. The lines corresponding to $z=1/2$ are the same as those found previously.

Solving (4) and (3′) simultaneously we find $2y/b=4r(2+3r^2)$, which has a maximum of $\sqrt{6}/3$ at $r=\sqrt{6}/3$ and has a value of $4/5$ at $r=1$ and $r=2/3$. So we may conclude that for $z=\sqrt{6}/3$ there is one line, and for $4/5 \leqq z \leqq \sqrt{6}/3$ there are two lines. The lines corresponding to $z=4/5$ are the same as those found previously.

Combining results we have:

value of z	number of lines
$0 \leqq z < \sqrt{3}/3$	2
$z = \sqrt{3}/3$	1
$\sqrt{3}/3 < z < (3+\sqrt{3})/6$	0
$z = (3+\sqrt{3})/6$	2
$(3+\sqrt{3})/6 < z < \sqrt{6}/3$	4
$z = \sqrt{6}/3$	3
$\sqrt{6}/3 < z \leqq 1$	2

Also solved by W. E. Brooke, Free Jamison, Roger Lessard, C. S. Ogilvy, and Prasert Na Nagara. Most of these solutions were incomplete.

Jamison pointed out that an interesting allied problem would be to determine through what part of the area of a triangle it is impossible to draw a line which will cut off one-third of the area of the triangle.

The Generalized Steiner-Lehmus Problem

E 863 [1949, 263]. *Proposed by W. O. Pennell, Exeter, N. H.*

If $\alpha>0, \beta>0, \alpha+\beta<\pi, 0<k<1$, then $\alpha=\beta$ is a necessary and sufficient condition for

$$\sin\alpha \sin(k\alpha+\beta) = \sin\beta \sin(k\beta+\alpha).$$

III. *Solution by D. J. Price, University of Malaya, Singapore.* Set $\theta=(\alpha+\beta)/2$ and $\phi=(\alpha-\beta)/2$ and substitute in the equation. One obtains

$$\sin(\theta+\phi) \sin[(1+k)\theta-(1-k)\phi] = \sin(\theta-\phi) \sin[(1+k)\theta+(1-k)\phi]$$

which simplifies readily to

$$\sin(1+k)\theta \cos(1-k)\phi \cos\theta \sin\phi = \cos(1+k)\theta \sin(1-k)\phi \sin\theta \cos\phi.$$

One trivial solution, $\theta=0$, is excluded by the condition $0<\theta<\pi/2$. On the other hand $-\pi/2<\phi<\pi/2$, and therefore there is a solution $\phi=0$, or $\alpha=\beta$. Apart from these we must have

$$[\tan(1+k)\theta]/\tan\theta = [\tan(1-k)\phi]/\tan\phi.$$

Now the expression on the right can take only values in the range 0 to $1-k$, whereas that on the left must lie outside the range 0 to $1+k$. These ranges cannot overlap, within the imposed limits, and therefore no non-trivial solution is possible.

Note: The two previous solutions [1950, 37] contain oversights. Solution I is invalidated because the equation is not an identity for all k within the given range. It must be shown that no values of k are possible for there to be any solution other than $\alpha=\beta$.

In solution II the expression $(k+2)(\beta+\alpha)/2$ can exceed π, thus making the left side of the final equation positive, and removing the contradiction used in the proof.

Space Analogue of a Theorem by Reynolds

E 894 [1949, 691]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let perpendiculars through vertex A of tetrahedron $ABCD$ to the faces ABC, ACD, ADB cut the circumsphere of $ABCD$ in B', C', D' respectively. Show that the volumes of the polyhedra A-$B'C'D'$-B, A-$B'C'D'$-C, A-$B'C'D'$-D are equal to that of $ABCD$.

Editorial Note. In the above problem the proposer intended to extend to the tetrahedron a property of the triangle given by B. Reynolds, *Nouvelles Annales de Mathématiques*, 1884, question 1501, p. 494. The analogy, however, is incorrect. A correct analogy, subsequently pointed out by the proposer, is to *draw perpendiculars to the edges AB, AC, AD of the tetrahedron to meet the circumsphere again in B', C', D' respectively. Then the volumes of the polyhedra A-$B'C'D'$-B, A-$B'C'D'$-C, A-$B'C'D'$-D are equal to that of the given tetrahedron.* To establish this it suffices to observe that B', C', D' are diametrically opposite B, C, D, respectively, and triangles BCD, $B'C'D'$ are congruent and in parallel planes.

A generalization of the corrected analogy is suggested by taking B', C', D' as the reflections of B, C, D in an arbitrary given point P. For example, P may be taken as the center of a quadric surface circumscribing $ABCD$, and the above analogy is then the case where the quadric surface is the circumsphere of $ABCD$.

Although the problem as originally stated is not true in general, it is certainly true if the tetrahedron $ABCD$ is *trirectangular* at A.

A Congruence Related to Wilson's Theorem

E 896 [1950, 34]. *Proposed by F. J. Duarte, Caracas, Venezuela*

If p is an odd prime show that

$$[1\cdot 3\cdot 5\cdots(p-2)]^2 \equiv (-1)^{(p+1)/2} \pmod{p}.$$

Solution by Kirk Stewart, College of Puget Sound. Since

$$1\cdot 3\cdot 5\cdots(p-2) \equiv (1-p)(3-p)\cdots(-4)(-2), \pmod{p}$$

it follows that

$$[1\cdot 3\cdot 5\cdots(p-2)]^2 \equiv (1)(-2)(3)(-4)\cdots(p-2)(1-p)$$
$$\equiv (p-1)!(-1)^{(p-1)/2} \pmod{p}.$$

But, by Wilson's theorem,

$$(p-1)! \equiv -1 \pmod{p},$$

so that

$$[1\cdot 3\cdot 5\cdots(p-2)]^2 \equiv (-1)(-1)^{(p-1)/2} \equiv (-1)^{(p+1)/2} \pmod{p}.$$

Also solved by A. W. Boldyreff, J. L. Brenner, D. H. Browne, B. B. Gresham, F. D. Grogan, B. F. Hadnot, H. T. LaBorde, H. E. Lee, Roger Lessard, Prasert Na Nagara, L. C. Mathewson, G. W. Medlin, Ingram Olkin, L. A. Ringenberg, Alex Rosenberg, P. J. Schillo, N. C. Scholomiti, N. T. Seely, Jr., R. E. Shafer, E. P. Starke, Alfred Sylvester, Margaret Willerding, and R. V. Andree's Number Theory Class at the University of Oklahoma.

M. S. Klamkin and J. D. Swift pointed out that this problem expresses a result due to C. F. Arndt, *Archiv Math. Phys.*, 2, 1842, pp. 32, 34–35, and is also a special case of a formula due to M. Vecchi, *Periodico di Mat.*, 16, 1901, pp. 22–24. See pp. 276–277 of Dickson's *History*.

Brenner also found the following related results: Let u be the number of quadratic non-residues of p between 1 and $p/2$ inclusive. If p is a prime of the form $8m-1$, then

$$1\cdot 3\cdot 5\cdots(p-2) \equiv (-1)^{u+1} \pmod{p},$$

and if p is a prime of the form $8m-5$, then

$$1\cdot 3\cdot 5\cdots(p-2) \equiv (-1)^{u} \pmod{p}.$$

Discarding Cards

E 898 [1950, 34]. *Proposed by N. S. Mendelsohn, University of Manitoba*

A pack of N cards is disposed of as follows. The top card is placed at the bottom of the deck, the next card is discarded, the third card is placed at the bottom, the fourth discarded, *etc.*, this process being carried on until there is only one card left. Let $f(N)$ be the position of this card, from the top, in the original arrangement of the deck. Prove that

$$f(N) = 2N - 2^{[\log_2 N]+1} + 1,$$

where $[x]$ denotes, as usual, the largest integer not greater than x.

Solution by Roger Lessard, École Polytechnique, Montreal. When the selected card is on top there must be 2^n cards in the pack, and conversely. This is easily seen from the fact that if we reverse the process, then for every card brought from bottom to top another card must first be brought in. Thus when the selected card makes a complete cycle the number of cards is doubled.

Now let there be $2^n \leqq N < 2^{n+1}$ cards in the pack. The selected card will first come on top when $N-2^n$ cards have been discarded to bring the pack down to 2^n, and the same number of cards have gone to the bottom. As $2(N-2^n)$ cards have been taken from the top, the position of the selected card at the beginning was then

$$f(N) = 2(N - 2^n) + 1 = 2N - 2^{n+1} + 1.$$

But $n \leqq \log_2 N < n+1$, whence $n = [\log_2 N]$, and the theorem is established.

Note that if $N = a_1 a_2 \cdots a_r$ in the scale of 2, $a_1 = 1$, then $f(N) = a_2 a_3 \cdots a_r a_1$.

Also solved by Prasert Na Nagara, P. J. Schillo, N. C. Scholomiti, N. T. Seely, Jr., and the proposer.

Editorial Note. What is the solution to the corresponding problem where every rth card is discarded? It would seem to depend on the solution of the difference equation

$$f(N) \equiv r + f(N-1), \bmod N,$$

where $f(1) = 1$ and $0 < f(N) \leqq N$. Does any theory exist for such types of difference equations?

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4405. *Proposed by Paul Erdös, Princeton University*

Let $x_1, x_2, \cdots$ be a sequence of real numbers, $x_i > c > 0$. We assume that they approach all their limit points from below. Then, for any A, there is only a finite number of solutions of the equation

$$A = \sum_{k=1}^{\infty} a_k x_k,$$

where the a_k are positive integers.

A special case occurs if $0 < x_1 < x_2 < \cdots$.

The case $x_k = 1 - 2^{-k}$ was communicated to the Proposer by R. A. Rosenbaum.

4406. *Proposed by H. J. Zimmerberg, Rutgers University*

Show that the most general linear differential equation

$$\sum_{i=0}^{n} a_i[p(x)]^i D^i y = f(x),$$

where $p(x)$ is of class C' and the a_i are constants with $a_n \neq 0$, which is reducible to the case of constant coefficients by a transformation of the independent variable is the generalized form of Euler's equation, wherein $p(x)$ is linear in x.

4407. *Proposed by N. A. Court, University of Oklahoma*

If four conics are circumscribed about the faces of a tetrahedron (T) in such a way that at each vertex of (T) the three tangents to the three conics passing through that vertex are coplanar, then the four conics lie on a quadric surface.

4408. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute*

Prove that $\Gamma^{(n)}(1)$ is positive or negative according as the integer n is even or odd.

4409. *Proposed by Fritz Herzog, Michigan State College*

Given a power series $f(x) = a_1x + a_2x^2 + a_3x^3 + \cdots$ with real coefficients. It is assumed that at least one of the coefficients a_n, $n \geqq 2$, is different from zero and that the radius of convergence R is positive or infinite. Show that there exist constants u_n, $0 < u_n < R$, such that $\sum_n (-1)^n u_n$ converges but $\sum_n (-1)^n f(u_n)$ diverges.

SOLUTIONS

A Circle Configuration

4313 [1948, 505]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Two perpendicular chords AB, CD of a circle (O) intersect at a point P inside the circle. There are eight circles (O_1), (O_2), (O_3), (O_4), (O_1'), (O_2'), (O_3'), (O_4') tangent to both chords and also to the circle (O), the first four exteriorly, the last four interiorly, and such that (O_i) and (O_i') lie in opposite quadrants formed by the given chords. (1) The sum of the squares of the distances O_1O_1', O_2O_2', O_3O_3', O_4O_4' is independent of the position of P, and the products $(O_1O_1')(O_3O_3')$ and $(O_2O_2')(O_4O_4')$ are equal. (2) The radical axes of the circles (O_1), (O_1'), (O_3), (O_3') taken in pairs and those of the circles (O_2), (O_2'), (O_4), (O_4') taken in pairs, intersect one another in thirty-six points of which twelve are on the circle (O), four of these coinciding with the vertices of the square whose diagonals are parallel to AB and CD.

*Solution by the Proposer.** We choose a rectangular coördinate system Oxy so that the equation of APB is $y = y_0$, and that of CPD is $x = x_0$, with $0 < x_0^2 + y_0^2 < r^2$, where r is the radius of circle (O). Let also r_i and r_i' be the radii of

* Translated and condensed by W. E. Byrne, Virginia Military Institute.

circles (O_i) and (O_i'), and let their centers be $O_i: (x_i, y_i)$ and $O_i': (x_i', y_i')$. To fix the figure in mind suppose that under the auxiliary coördinate system PXY with

$$x = x_0 + (X - Y)/\sqrt{2}, \qquad y = y_0 + (X + Y)/\sqrt{2},$$

the XY coördinates of O_1, O_2, O_3, O_4, O_1', O_2', O_3', O_4' are:

$$(0, r_1\sqrt{2}), \quad (r_2\sqrt{2}, 0), \quad (0, -r_3\sqrt{2}), \quad (-r_4\sqrt{2}, 0),$$
$$(0, -r_1'\sqrt{2}), \quad (-r_2'\sqrt{2}, 0), \quad (0, r_3'\sqrt{2}), \quad (r_4'\sqrt{2}, 0).$$

To determine, for example, (x_2, y_2) and r_2, we have

$$r_2 = x_2 - x_0, \qquad r_2 = y_2 - y_0, \qquad x_2^2 + y_2^2 = (r + r_2)^2.$$

Elimination of x_2 and y_2 gives a quadratic whose positive root is

$$r_2 = r - x_0 - y_0 + \sqrt{2[r^2 - (x_0 + y_0)r + x_0y_0]}.$$

Similarly,

$$r_1 = x_0 - y_0 + r + \sqrt{2[r^2 + (x_0 - y_0)r - x_0y_0]},$$
$$r_1' = -x_0 + y_0 - r + \sqrt{2[r^2 + (x_0 - y_0)r - x_0y_0]},$$
$$r_2' = x_0 + y_0 - r + \sqrt{2[r^2 - (x_0 + y_0)r + x_0y_0]}.$$

The calculations for the other circles are of the same type. From them we obtain

$$\overline{O_1O_1'}^2 = 16[r^2 + (x_0 - y_0)r - x_0y_0],$$
$$\overline{O_2O_2'}^2 = 16[r^2 - (x_0 + y_0)r + x_0y_0],$$
$$\overline{O_3O_3'}^2 = 16[r^2 - (x_0 - y_0)r - x_0y_0],$$
$$\overline{O_4O_4'}^2 = 16[r^2 + (x_0 + y_0)r + x_0y_0].$$

Hence we have

$$\overline{O_1O_1'}^2 + \overline{O_2O_2'}^2 + \overline{O_3O_3'}^2 + \overline{O_4O_4'}^2 = 64r^2,$$

which is independent of the position of P, and

$$\overline{O_1O_1'}^2 \cdot \overline{O_3O_3'}^2 = 256(r^2 - x_0^2)(r^2 - y_0^2) = \overline{O_2O_2'}^2 \cdot \overline{O_4O_4'}^2.$$

For the second part of the problem it is convenient to designate by $L(1, 1')$, $L(1, 3)$, $L(2, 4)$, etc., the radical axes of the corresponding pairs of circles. There are six radical axes for the circles (O_2), (O_4), (O_2'), (O_4') taken in pairs and six radical axes for the circles (O_1), (O_3), (O_1'), (O_3') taken in pairs. Their intersections give 36 points. A clue as to how to match them to obtain the twelve points on (O) may be had by considering the limiting case where P coin-

cides with O, as the figure is simple to sketch and continuity enables us to reduce the number of calculations.

We find that $L(1, 1')$, $L(3, 3')$, $L(2, 2')$, $L(4, 4')$ meet in the vertices of a square inscribed in (O), which is also true for P coincident with O. The remaining eight points cited as being on (O) are obtained from the two sets:

$$L(2, 4), \quad L(2', 4'), \quad L(1, 3'), \quad L(1', 3);$$
$$L(2', 4), \quad L(2, 4'), \quad L(1, 3), \quad L(1', 3').$$

To save space a process of verification is indicated without giving the details of all the calculations. To find, for example, the intersection of $L(1, 1')$ and $L(2, 2')$ we note the equations

$$L(1, 1'): \quad Y = (r_1 - r_1')/2\sqrt{2},$$
$$L(2, 2'): \quad X = (r_2 - r_2')/2\sqrt{2},$$

which, with the values for the radii as previously given, show that the intersection (X, Y) is $(0, r)$.

Also solved by Joseph Langr.

Editorial Note. The Proposer notes that the analogous proposition for three dimensional space is true, using a sphere with three mutually perpendicular planes through an interior point. Langr refers to a previous solution of his, published in *Mathesis*, VIII, 1908, question 1660, p. 173.

Area Within Simple Closed Plane Curve

4320 [1948, 641]. *Proposed by V. F. Ivanoff, San Francisco*

Let $f(s, \phi) = 0$ be the intrinsic equation of a simple closed curve; s is the length of arc of the curve, measured from a fixed point P_0 to any point P, and ϕ is the angle between the tangent lines at P and P_0. Find the area enclosed by the curve.

Solution by Mary Payne, Michigan State College. Construct a Cartesian coordinate system with the origin at P_0, x-axis in the direction of the slope of the curve at P_0, and y-axis directed so that s is measured from P_0 in the direction of the positive x-axis. Then ϕ is numerically equal to the inclination of the tangent. Let us take ϕ, however, to be the angle between the tangent directed in the direction of increasing s and the positive x-axis. Then ϕ will take on all values between 0 and 2π. We regard $f(s, \phi) = 0$ as being solved for $\phi = \phi(s)$.

Now we take a rectangular strip parallel to the x-axis and between the y-axis and a point on the curve at a distance s from the origin (measured along the curve). The base of this strip is $\int_0^s \cos \phi(t)dt$, and the altitude is $\sin \phi(s)ds$. Hence the total area is

$$\int_0^L \sin \phi(s) \left[\int_0^s \cos \phi(t)dt \right] ds,$$

where L is the total length of the curve.

The correctness of this formula is obvious for a curve which is always convex, and nearly obvious if any line parallel to the x-axis intersects the curve in at most two points. If, however, a line parallel to the x-axis intersects the curve in more than two points, the interpretation of ϕ given above is necessary to make the formula true. The area will be positive if $\phi > 0$ for small s and negative if $\phi < 0$ for small s.

Also solved by C. S. Ogilvy and the Proposer.

Editorial Note. Ogilvy obtained the formula

$$\frac{1}{2}\int_s \left\{\sin\phi \int \cos\phi ds - \cos\phi \int \sin\phi ds\right\} ds,$$

which is a direct consequence of the familiar theorem

$$A = \frac{1}{2}\int_s (x dy - y dx).$$

Divisibility in a Sequence of Integers

4330 [1949, 40]. *Proposed by Paul Erdös, Syracuse University*

Let $a_1 < a_2 < \cdots$ be an infinite sequence of integers. Prove that there exists either an infinite subsequence in which no integer divides another or an infinite subsequence where each integer is a multiple of the preceding one.

I. *Solution by R. S. Lehman, Stanford University.* In the given sequence consider all integers each of which divides no other integer in the sequence. If the number of these is infinite, they are the required subsequence having the property that no member divides any other. If they are finite in number, remove them and all of their divisors from the sequence. The remaining integers will then form an infinite sequence in which each integer divides at least one other in the sequence. From this sequence we can select an infinite subsequence in which each integer divides the following. If b_k is chosen to be an integer in the subsequence, we can select any multiple of b_k appearing in the sequence to be b_{k+1}.

II. *Solution by G. A. Hedlund, Yale University.* C. Visser (*Proceedings of the Royal Academy of Amsterdam*, 1937, p. 359) has proved the following theorem: *Let there be given an infinite sequence of arbitrary elements. Suppose that in every infinite subsequence there exists at least one system of n different elements that has a certain property P. Then there exists an infinite subsequence in which every system of n different elements has the property P.*

To solve the proposed problem, suppose that in every infinite subsequence of the given sequence of integers $a_1 < a_2 < \cdots$ some integer divides some other. Apply Visser's Theorem with $n = 2$ and with two integers said to have property P if and only if one divides the other. We infer that there must exist a subsequence such that each b_i divides b_{i+1}.

III. *Solution by R. C. Buck, University of Wisconsin.* The proposed conclusion follows at once from a simple result of lattice theory. The positive integers form a lattice L under divisibility. A subset S of L is called free if no distinct pair of elements of S is ordered. Rephrasing the proposed problem, we must show that any infinite subset A of L contains either an infinite chain or an infinite free subset. This is true of any partially ordered infinite set A in which the set of all elements below any element is finite. The proof is as follows. Suppose that A has no infinite free subset. Choose any maximal free subset A_0 of A. This is a finite set, and every other element of A is above or below some element of A_0. Hence, there is an element α_0 of A_0 such that infinitely many elements of A lie above it. From these, choose a maximal free subset A_1; this is again finite, and contains an element α_1 above which are infinitely many elements of A. Proceeding thus, we generate an infinite chain $\alpha_0 < \alpha_1 < \cdots$.

Also solved by Harley Flanders, John Kelly, Burnett Meyer, and N. L. Rabinovitch.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 115th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Intermediate Algebra for Colleges. By P. R. Rider. The Macmillan Company, New York, 1949. 10+242 pages. $2.75.

In the preface the author states, "This book is designed for college students who have had one year of high school algebra. It is considerably simpler than Rider's *College Algebra*, but the methods of presenting the material are those which have proved successful in that book. The explanations are set forth principally by the use of illustrative examples which bring out the principles involved. The student can follow these examples much more easily and with a better understanding than he can read more abstract discussions. If he works through them carefully he grasps the meaning of the underlying mathematics with greater facility. This is particularly true in an elementary course.

"A summary is provided at the end of each chapter. It is hoped that these summaries will prove helpful to the student.

"There is a liberal supply of carefully chosen exercises. These are not too difficult, yet some of them offer a challenge to the brighter student. Answers to the odd-numbered exercises are to be found at the back of the book, answers to the even-numbered are printed in a separate pamphlet."

The author has carried out his intentions very well. The definitions are carefully stated, duly noting the restrictions. The illustrative examples are numer-

ous and well chosen, but perhaps the outstanding feature of the book is the wealth of well graded exercises, and the excellent and abundant collection of worded problems. The exercises at the end of the chapter on ratio, proportion, and variation are especially good. The summaries of definitions and procedures which appear at the end of each chapter should be very useful to the students.

This reviewer would have welcomed an introductory chapter with a large selection of simple exercises on permutations, combinations, and probability. Such an addition would have enhanced the value of the book for pre-medical students, as well as have provided a use for the included chapter of five pages on the binomial formula.

This carefully written book is undoubtedly one of the best in its field. The excellent printing and the balanced and uncrowded arrangement of the material on each page contributes greatly to the attractiveness of the book.

K. C. SCHRAUT

Techniques of Statistical Analysis. By the Statistical Research Group, Columbia University, Applied Mathematics Panel, Office of Scientific Research and Development. Edited by Churchill Eisenhart, M. W. Hastay, and W. A. Wallis. New York and London, McGraw-Hill Book Company, Inc., 1947. 14+473 pages. $6.00.

According to the preface, written by W. A. Wallis, Director of Research, the Statistical Research Group at Columbia University was organized July 1, 1942, and was disbanded soon after the end of the war. The senior scientific staff consisted of seventeen members and Wallis says that each of these, together with five members of the junior scientific staff, had an influence on the book. Therefore, the general excellence of the work is not surprising and it is difficult for this reviewer to find any points for criticism.

In order to help the Army, Navy, Office of Scientific Research and Development, and others, with their statistical problems, it was necessary for various members of the Statistical Research Group to investigate the merits of existing statistical techniques and, in some cases, to develop new ones. This book serves as a record of a part of that work. Its complete title, *Selected Techniques of Statistical Analysis for Scientific and Industrial Research and Production and Management Engineering*, properly describes the nature of the volume and the areas in which it could be most useful.

The seventeen thought-provoking chapters of the book were prepared by nine different authors. The chapters vary in length from eight to ninety-two pages and each treats a different topic. However, the book is surprisingly well unified because of exceptionally fine work by the editors and others responsible for its form.

Each chapter is preceded by a table of contents and ended by a good list of references. Notation is uniform and care has been taken in recording its meaning. Figures and tables follow a uniform pattern. Early in each chapter there is a

description of the type of practical problem which might have given rise to the investigation under discussion and, in many of the chapters, there are detailed solutions for specific examples. The book closes with an unusually complete index. No effort has been spared to put the content into as usable a form as possible.

The book has a few rather obvious misprints. The proof of one theorem (pp. 409–10) seems faulty. This reviewer does not agree with the author of chapter three regarding the reason for the use of range in statistical quality control. However, these are minor matters and do not alter the reviewer's opinion that this is an excellent book.

E. G. OLDS

Solid Analytic Geometry. By Adrian Albert. New York, McGraw-Hill Book Company, Inc., 1949. 10+162 pages. $3.00.

The author's apparent chief aim in this book, namely, to present the study of analytical geometry of space by means of the theory of vectors and matrices is a laudable one. He begins the book by an exposition of vectors in n-space and obtains as consequences the usual formulas concerning lines and planes in ordinary space. In the third chapter he gives an elementary treatment of curves and surfaces, and this is followed in the fourth chapter by a brief treatment of spheres. The fifth chapter, which covers the usual discussion of quadric surfaces and their classification, is followed by a useful and elegantly written chapter on the theory of matrices which is necessary for the reading of the remainder of the book. This chapter treats solutions of systems of linear equations, the characteristic equation of an n-rowed square matrix, and the orthogonal reduction of a symmetric matrix. With this background an excellent presentation of rotation of axes and the orthogonal reduction of a real quadratic form are afforded in the seventh chapter.

Some of the material of Chapter 8 on spherical coordinates is novel for a textbook on solid geometry. It contains the development of formulas for parallax corrections in the spherical coordinates incident to translation of the rectangular axes. Planar rotations are named by the terms pitch, roll, or yaw according as the x, y, or z axis is left invariant in the rotation. A general rotation matrix is shown to be expressible as the product of a pitch matrix, a roll matrix, and a yaw matrix, and the relations of the elements of the rotation matrix to the angles of the planar rotations are given. Gnomonic projections and charts are treated in the remainder of the chapter.

In the final chapter a brief discussion of projective geometry is given by the use of matrices. Besides consideration of invariant points, a concise and rigorous demonstration of the invariance of the cross ratio of four points under a projective transformation is included.

The reviewer is of the opinion that the chief fault of the volume, as a textbook on solid analytical geometry, lies in the paucity of stimulating problems on geometry. For instance, merely finding a sphere through four points is not much

fun, but arriving at necessary and sufficient conditions under which two circles lie on the same sphere, or working out the locus of the center of a sphere with one degree of freedom would afford more interest and information for the student. The triviality of many sets of exercises could be reduced by the introduction of a few good problems which would require more of the student than mere substitution of numbers into formulas or routine computation. The word locus is not mentioned in the book. At least, it is not in the index! A set of answers to exercises is not supplied.

The printing and format are excellent. The following errors or misprints were noticed. On page 27, the peculiar statement $(a, b, c) = (0, 0, 0)$ appears. Read z for c in $f(x, y, c)$ on page 32. On page 36, reference should be to (22) instead of (21). One should read "consisting *of* all points" on page 43. The x in the denominator in equations (2) on page 54 should be read as a.

This is a useful addition which all teachers of solid analytical geometry should read and assimilate. Whether they should use it as a textbook depends upon their point of view. Certainly, the student can gain much valuable information in algebra by use of the book.

C. E. SPRINGER

On the Theory of Stochastic Processes and Their Application to the Theory of Cosmic Radiation. By N. Arley, New York, Wiley, 1949. 240 pages. $5.00.

Let $x(t)$ be for each time $t \geqq 0$ a random variable which can take on only integral values. Suppose that the (conditional) probability of a transition from m at time r to n at time $s > r$ is the same as the conditional probability of this transition if $x(t)$ is also prescribed at times before r. In physical language we speak of a system which at time t is in some state denoted by an integer $x(t)$; the system jumps from state to state; the distribution of states after a given jump depends on the state just before the jump, but not on previous states. Analytically the process is determined by the probability $p_{mn}(r, s)$ of a jump to state n at time s from the state m at time r. If $P(r, s)$ is the matrix of these transition probabilities it is shown that $P(r, t) = P(r, s)P(s, t)$ if $r < s < t$, and this functional equation dominates the analytical work.

Processes of this type have been applied in many ways in recent years. For example $x(t)$ can be taken to be the number of radioactive disintegrations of atoms of some substance that have occurred between times 0 and t; $x(t)$ can be taken as the number of individuals (say bacteria or elementary physical particles) at time t in a population which grows by fission as modified perhaps by deaths; $x(t)$ can be taken as the fortune (in dollars) at time t of a stock market speculator or the salary (in cents) at time t of a university professor. In each case of course the situation must be analyzed to check the appropriateness of the mathematical model.

The matrix $A(s)$ with $a_{mn}(s) = (\partial/\partial t)\ P(s, t)$ (for $t = s$) has a simple physical interpretation; $a_{mn}(s)ds$ is for $m \neq n$ the probability at time s of a transition from m to n in time ds and $1 + a_{mm}(s)ds$ is the probability of remaining at m. In many

applications the matrix $A(s)$ is known from theoretical considerations or can be estimated from observations, and the transition probabilities $P(s, t)$ can then be determined, for example, by solving the system of differential equations

$$\frac{\partial p(s, t)}{\partial s} = -A(s)P(s, t)$$

in s for fixed t, or the system

$$\frac{\partial p(s, t)}{\partial t} = P(s, t)A(t)$$

in t for fixed s. In both cases the initial conditions are determined by the fact that $A(r, r)$ should be the identity matrix.

The author gives a survey of these stochastic processes, examining in detail the best known special cases: the Poisson process in which only jumps from m to $m+1$ are possible and $a_{mm+1}(s) \equiv$const.; the birth process suggested by individuals dividing independently of each other at random times into pairs which then further divide, and so on, so that the number of individuals at time t can again only jump from m to $m+1$, and $a_{mm+1}=$const. m; the Polya process, which contains both of these as special cases, and for which again only jumps from m to $m+1$ are possible with $a_{mm+1}(s) = \lambda(1+bm/1+b\lambda t)$ where b and λ are preassigned constants. Multi-dimensional processes, in which $x(t)$ is a vector, are also discussed.

The strictly mathematical section occupies only about one third of the book. The rest of the book contains the application of these stochastic processes to cosmic ray cascade showers, in which moving elementary particles generate other particles in the course of their motion, the new particles generate more, and so on, until the process dies out as the particles lose energy. Several stochastic models of this process have been proposed, all of the type described above. The author analyzes these models and proposes another, together with a computational technique. The difficulty of the problem is increased by the fact that only the crudely unrealistic models lend themselves to easy numerical computation.

This is not the place to evaluate the author's contribution to physics, even if the reviewer were qualified to do so. In any event, the book, which is the author's doctor's thesis, contains a useful discussion of special stochastic processes of the type described above (Markov processes with a finite or enumerable state space). These processes arise in many applications, and most of the material on them has not been readily available. The book's coverage of these processes is not exhaustive, and does not claim to be. The stress is on the application to the cascade problem.

J. L. Doob

Correction. In the May, 1950, issue of this MONTHLY the number of pages in *Plane and Spherical Trigonometry* by M. Richardson was erroneously given as 15+138 pp. It should have read 15+343 pages.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS 1948–1949

Mathematics Club, Oberlin College

The *Mathematics Club* of Oberlin College had a very successful year, the high spots of which were the Christmas Party and the Annual Banquet. Incorporated into the program were several talks by various professors, outside the Mathematics Department, on the importance of mathematics in their respective fields. The speakers and their topics were:

Mathematics and logic, by Joseph Solomon
Magic squares, by Benson Scheff
Paper-folding geometry, by Anne Johnson
Micro-wave optics, by Prof. Wade Ellis
Unicursal curves, by Marjorie Ireland
Poincare's model of hyperbolic geometry, by Robert Keesey
Mathematical molecules, by Prof. J. A. Campbell
Seventeen-sided polygon, by William Cook
Nine-point circle, by Janet Brown
Transformations and topology, by Emery Thomas
Mathematical calculators, by Jeanne Taylor
Theory of finite groups, by William Kossler
The golden section, by Patricia Mott.

Prof. F. W. Reed of Ohio University spoke on the topic of *Linear sines and cosines* at the Annual Banquet.

The officers during the year 1948–49 were: President, Emery Thomas; Vice-president, Joseph Solomon; Secretary-Treasurer, Evelyn Schmidt and Benson Scheff; Social Chairman, Anne Johnson; Publicity Chairman, Jeanne Taylor; Faculty Advisor, Prof. R. W. Wagner.

Kappa Mu Epsilon, Washburn Municipal University

The *Kansas Delta* Chapter of *Kappa Mu Epsilon* meets regularly on the second Thursday of each month. Some of the special topics during the school years were:

Some famous problems in mathematics, by Prof. G. B. Price
The part that Kappa Mu Epsilon may play in the rehabilitation of patients at Winter General Hospital, by Dr. H. H. Wagenheim, of the Menninger Foundation
Probability, by Mr. T. D. McAdam
Logic and finite geometry, by Mr. K. E. Lake
Some inequalities, by Dr. Robert Schatten

An Initiation Dinner and other social meetings were held during the year.

In April, the *Kansas Delta* Chapter was host to the National Biennial Convention of *Kappa Mu Epsilon* held in Topeka.

The officers for the school years 1949–50 are: President, L. L. Johnson; Vice-president, Mary McCullough; Secretary, Donna Lee Simmons; Treasurer, Steve Powell; Sponsor, Miss Laura Z. Greene; Corresponding Secretary, Miss Margaret E. Martinson.

Mathematics Club, Queens College

Papers presented to the Queens College *Mathematics Club* during 1948–49 were:

Quaternions, by Prof. J. M. Feld

Statics in n-space, Prof. Leon Cohen

Surfaces, By Nick Roescher

Fields and differential fields, by Prof. H. W. Raudenbush, Jr.

Words, by Miss Jean Hirsh.

Officers for last term were: President, Nick Rescher; Vice-president, Gertrude Nissenbaum; Secretary, Dolores Lehrman; Sponsor, Mr. A. B. Brown.

The Mathematics and Physics Club, College of St. Thomas

Papers presented to the *Mathematics and Physics Club* of the College of St. Thomas during the year 1949–50 included:

Topology, by Dr. G. K. Kalisch, Professor of Mathematics, University of Minnesota

Energy production in the stars, by Dr. G. Blass, Professor of Physics

Effects of the atom bomb on human beings, by Dr. J. Ryan, Chief of Medicine, St. Joseph's Hospital, St. Paul, Minnesota.

Other activities of the Club included attendance at a meeting of the Mendel Forum Club, of St. Catherine's College, at which Mr. F. A. Gifford, Meteorologist at Northwest Airlines, spoke on *The effect of climate on certain aspects of human life.*

A number of films were also shown during the year.

The officers for the year 1949–50 are: President, Stephen Dickenson; Vice-president, Richard Zemlin; Secretary, Clarence Germain; Treasurer, Marcus McEllistrem; Moderator, Dr. Laurence Sheridan.

Pi Mu Epsilon, Carnegie Institute of Technology

Some of the papers presented at the monthly meetings of the *Pennsylvania Epsilon* Chapter of *Pi Mu Epsilon* during 1948–49 were:

Classical problems in geometry, by Prof. J. H. Neelley.

Complex numbers and quaternions, by Visiting Professor F. D. Murnaghan of Carnegie Institute of Technology and Professor at the Istituto Technico Aeronautica of Brasil.

Science and art in the Italian renaissance, by Visiting Professor Enrico Bom-

piani of the University of Pittsburgh and Professor at the University of Rome.

Mathematical logic, by Visiting Professor E. R. Lorch of Carnegie Institute of Technology and Professor at Columbia University.

At the annual banquet, thirty-seven persons were initiated, of which thirteen were undergraduates, eight were graduates, and sixteen were faculty members.

The officers for 1949–50 are: Director, R. C. DiPrima; Vice-Director, D.H. Shaffer; Secretary; W. H. Warner; Treasurer, F. B. Smith; Faculty Adviser, Prof. J. B. Rosenbach; Executive Committee, Richard Cutkosky, Robert McKelvey, and William Simon.

Mathematics Club, Collinsville, Illinois, Township High School

Titles of talks given before the Collinsville Township High School *Mathematics Club* include:

Transfinite numbers—a discussion of infinity, by Dick Stowe

Ancient methods of multiplication, by Dorothy Keck

Solutions of some problems from the Stanford University Mathematics Competition for high schools, by Don Bitzer

Euler's totient, by Mel Lieberstein

Some mathematical puzzles and jokes, by Fred Malone

The four-color problem and its extension to the torus, by George Rupprecht

The use of mathematics in music, by Earl Cooley.

The Club sponsors a "Problem of the Week" bulletin board and has succeeded in getting the entire school interested in many of the problems proposed there. It also sponsors weekly chess nights.

The officers are: President, Douglas Phillips; Vice-president, Ivan Vurgener; Secretary, Ethel Rowland; Sponsor, Mel Lieberstein.

Mathematics Club, University of Dayton

Biweekly meetings were held at which members of the Club presented the following papers:

The exact value of pi, by Daniel J. Groszewski

The history and computation of pi, by Robert Hennessey

The application of differential equations in chemical kinetics, by George Moon

Interpolation, by Thomas Riney

The solution of a right triangle without the aid of trigonometric tables, by George Oberer

The set of irrational numbers, by Demetrius Zonars

The mechanics of impact, Richard Segers

The paradox and its place, by Donald Moore

Methods of summation by the calculus of finite differences, by Thomas Beckert

Mathematical probability applied to games of chance, by Richard Biedenbender.

Faculty papers presented during the year were:

Points of intersection of concentric circles, by Dr. K. C. Schraut

How much does a can of tomatoes weigh?, by Prof. C. G. Peckham

Continued fractions and matrices, by Prof. J. S. Frame of Michigan State College.

Several excursions were taken to the University of Cincinnati and the club was host to the Annual Colloquium of the National Mathematics Honor Society for Secondary Schools.

Dr. Frame spoke at the annual dinner at which time an award was given to Thomas Beckert for having presented the most interesting paper during the second term.

Officers during the year were: President, George Moon; Vice-President, John Quinlisk; Secretary, Joseph Schell; Treasurer, Jorge Nunez; Publicity Secretary, Richard Segers and Robert Hennessey.

Kappa Mu Epsilon, Kansas (Pittsburg) State Teachers College

Kansas Alpha chapter of *Kappa Mu Epsilon* held four open meetings at which the following talks were presented:

Mathematics and civilization, by Prof. J. A. G. Shirk

Inevitability of scientific discovery, by Norval Phillips

Development of commercial mathematics, by Mary Morrison

Reports of Kappa Mu Epsilon convention, by Everett Ard and James Helmert

Report of National Council meeting, by Helen Kriegsman.

A combined meeting with the *Physical Science Club* was held for viewing industrial films on the *Production and uses of electric power*, and on *Special uses of glass in electric insulation.*

Three Kappa Mu Epsilon keys were awarded to the three seniors making the highest scholastic record in mathematics. They were awarded to Norval Phillips, Mary Dell Morrison, and James Pike.

Officers for 1948–49 were: President, Norval Phillips; Vice-President, Everett Ard; Secretary, Mary Dell Morrison; Treasurer, William Sellers; Corresponding Secretary, Prof. J. A. G. Shirk; Faculty Sponsor, Dr. R. G. Smith.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PRELIMINARY ACTUARIAL EXAMINATIONS, PRIZE AWARDS

The winners of the prize awards offered by the Society of Actuaries to the

nine undergraduates ranking highest on the score of Part 2 of the 1950 Preliminary Actuarial Examinations are as follows:

First Prize of $200

Mattuck, Arthur P., Swarthmore College

Additional Prizes of $100

Dempster, Arthur P., University of Toronto
Haslam, M. Brent, The University of Buffalo
Hudek, Paul R., University of Minnesota
Jamieson, J. Rae, University of Toronto
Leff, Milton M., The University of Western Ontario
Milnor, John W., Princeton University
Reynolds, William F., College of the Holy Cross
Walter, John R., University of Toronto

The Society of Actuaries has authorized a similar set of nine prizes for the 1951 examinations on Part 2.

The Preliminary Actuarial Examinations consist of the following three examinations:

Part 1. Language Aptitude Examination.
(Reading comprehension, meaning of words and word relationships, antonyms, and verbal reasoning.)

Part 2. General Mathematics Examination.
(Algebra, trigonometry, coordinate geometry, differential and integral calculus.)

Part 3. Special Mathematics Examination
(Finite differences, probability and statistics.)

The 1951 Preliminary Actuarial Examinations will be prepared by the Educational Testing Service and will be administered by the Society of Actuaries at centers throughout the United States and Canada on May 18, 1951. The closing date for applications is March 15, 1951.

Detailed information concerning the Examinations can be obtained from: The Society of Actuaries, 208 South LaSalle Street, Chicago 4, Illinois.

FIRST NATIONAL CONGRESS OF APPLIED MECHANICS

The First National Congress of Applied Mechanics will be held on June 11–16, 1951, at Illinois Institute of Technology, Chicago, Illinois. The general chairman of the Congress is Professor L. H. Donnell of Illinois Institute of Technology.

Papers for presentation at the Congress must be submitted before April 14, 1951, and should constitute original research in applied mechanics, which includes kinematics, dynamics, vibrations, waves, mechanical properties of materials and failure; stress analysis elasticity, plasticity; fluid mechanics; thermodynamics. Papers should not be longer than 5,000 words. They will be grouped by subject and one half hour will be allotted for presentation and discussion of each paper.

PERSONAL ITEMS

The John Simon Guggenheim Memorial Foundation has granted awards in mathematics to the following: Dr. J. H. Bigelow, Institute for Advanced Study; Professor Samuel Eilenberg, Columbia University; Associate Professor N. E. Steenrod, Princeton University; Dr. R. P. Boas, Jr., executive editor of *Mathematical Reviews*, Cambridge, Massachusetts; and Associate Professor Philip Hartman, Johns Hopkins University.

Mr. C. J. A. Halberg, Jr., who has been an instructor in mathematics during the past year at Pomona College, has been awarded the Honnold Fellowship by Pomona College.

Professor S. B. Jackson of the University of Maryland represented the Association at the Conference on Cultural and Educational Relations with Japan held at Washington on May 25, 1950, under the auspices of the Commission on the Occupied Areas.

Alabama Polytechnic Institute announces the following: Assistant Professor J. C. Eaves, University of Alabama, has been appointed to an associate professorship; Dr. Ernest Ikenberry of Louisiana State University, Dr. Nathaniel Macon of the University of North Carolina, Dr. A. J. Owens, University of Florida, and Dr. W. A. Rutledge, University of Tennessee, have been appointed to assistant professorships; Mr. H. W. Burnette and Mr. R. K. Butz of the University of Georgia, Mr. J. G. Cox, Alabama Polytechnic Institute, and Mr. V. E. Dietrich, Purdue University, have been appointed to instructorships.

Colorado College makes the following announcements: Professor W. V. Lovitt, head of the Department of Mathematics has retired; Assistant Professor J. S. Leech of the University of Chicago has been appointed Professor of Mathematics and Head of the Department of Mathematics.

Denison University reports: Professor F. B. Wiley, chairman of the Department of Mathematics, has retired with the title of Professor Emeritus and has accepted the position of Chairman of the Mathematics Department of Ashland College; Associate Professor Chosaburo Kato has been promoted to the position of Professor and Chairman of the Department of Mathematics; Assistant Professor Marion Wetzel has been promoted to an associate professorship.

De Paul University announces: Dr. John DeCicco of Illinois Institute of Technology has been appointed Professor and Chairman of the Department of Mathematics; Dr. W. B. Caton, Washington State College, has been appointed to an assistant professorship.

Oberlin College announces the following: Assistant Professor Wade Ellis and Assistant Professor E. P. Vance, chairman of the Department of Mathematics, have been promoted to associate professorships.

At Pratt Institute: Mr. F. S. Beckman and Mr. R. J. Kohlmeyer have been promoted to assistant professorships.

Reed College and Swarthmore College announce the following exchange of professors: Professor R. A. Rosenbaum of Reed College will be a member of the faculty of Swarthmore College during the year 1950–1951 and Assistant Pro-

fessor P. W. Carruth of Swarthmore College will join the faculty of Reed College for the year.

Rutgers University announces the following promotions: Associate Professors H. S. Grant, E. R. Ott, M. S. Robertson to professorships; Assistant Professor C. R. Phelps to an associate professorship; Dr. R. M. Cohn to an assistant professorship.

University of Arizona announces the following appointments: Dr. D. G. Duncan, University of Michigan, and Dr. L. E. Payne, Iowa State College, to assistant professorships; Mr. B. R. Cato, Jr. of Duke University and Mr. V. A. Kramer, University of Chicago, to instructorships.

At the University of Delaware: Professor G. C. Webber has been promoted to the position of Chairman of the Department of Mathematics; Assistant Professor R. F. Jackson has been promoted to an associate professorship.

Mr. M. R. Bates of Union College has accepted a position as mathematician at the Bell Aircraft Company, Buffalo, New York.

Dr. R. P. Boas, Jr., executive editor of *Mathematical Reviews*, has been appointed to a professorship at Northwestern University.

Assistant Professor H. K. Brown of Northeastern University has been promoted to the position of Associate Professor of Mechanical Engineering and has been appointed Director of Engineering Graduate Study.

Professor Lamberto Cesari is on leave from the University of Bologna and has been appointed to a visiting professorship at Purdue University.

Associate Professor C. H. Denbow of the United States Naval Postgraduate School has been appointed to a professorship at Ohio University.

Professor Nat Edmonson, Jr., of Johns Hopkins University has a position now with the Fairchild Engine and Aircraft Corporation, Oak Ridge, Tennessee.

Mr. G. W. Evans, II, of New York University has accepted a permanent position with the Argonne National Laboratory, Chicago, Illinois, as an associate mathematician.

Professor William Feller of Cornell University has been appointed Eugene Higgins Professor of Mathematics at Princeton University.

Dr. Sze-tsen Hu has been appointed to an assistant professorship at Tulane University; during the academic year 1950–1951 he will be on leave of absence and will be a member of the Institute for Advanced Study.

Associate Professor F. Burton Jones of the University of Texas has been appointed to a professorship at the University of North Carolina.

Professor E. S. Kennedy of the American University of Beirut, Lebanon, is on leave and has accepted a Rockefeller Fellowship at the Institute for Advanced Study.

Professor W. I. Layton of State Teachers College, Frostburg, Maryland, has been appointed Head of the Department of Mathematics of Stephen F. Austin State Teachers College, Nacogdoches, Texas.

Professor J. C. C. McKinsey of Oklahoma Agricultural and Mechanical College has accepted a position as mathematician at the Rand Corporation, Santa Monica, California.

Professor L. L. Merrill of Clarkson College of Technology has been promoted to the position of Chairman of the Department of Mathematics; he is also serving as a consultant for the Stromberg Carlson Company, Rochester, New York.

Miss Elsie C. Muller has been appointed to an instructorship at LaSalle-Peru Township High School and Junior College, LaSalle, Illinois.

Dr. J. H. Mulligan, Jr., of Allen D. DuMont Laboratories, Inc. has been appointed to an assistant professorship of electrical engineering at New York University.

Professor W. R. Murray of Franklin and Marshall College has been appointed to an associate professorship at Wagner College.

Associate Professor Ivan Niven of the University of Oregon has been promoted to a professorship.

Assistant Professor G. G. Roberts of Berea College has been promoted to an associate professorship.

Professor J. B. Rosenbach, assistant head of the Department of Mathematics, Carnegie Institute of Technology, has been promoted to the position of Head of the Department of Mathematics.

Professor A. C. Schaeffer of Purdue University has been appointed to a professorship at the University of Wisconsin.

Mr. D. J. Smith of the University of Notre Dame has accepted a position with the Royal-Liverpool Group of Insurance Companies, New York, New York.

Mrs. Dorothy M. Stone of Manchester, England, has been appointed Mary Whiton Calkins Visiting Professor of Mathematics at Wellesley College for the year 1950–1951.

Mr. H. J. Vandort is now Assistant Rector, Grace Episcopal Church, Grand Rapids, Michigan.

Professor Abraham Wald, chairman of the Department of Mathematical Statistics, Columbia University, gave a series of lectures on the theory of statistical decision functions at the Naval Ordnance Test Station, April 3–7, 1950.

Dr. J. V. Wehausen has been appointed Executive Editor of *Mathematical Reviews*.

Professor R. L. Wilder of the University of Michigan was visiting lecturer at the University of Oregon for the week May 15–20, 1950.

Professor Constantin Carathéodory of Munich, Germany, died February 2, 1950.

Mr. B. H. Lowe of the University of Akron died on April 24, 1950.

Professor Raleigh Schorling, University of Michigan, died on April 22, 1950.

Professor J. A. G. Shirk of Kansas State Teachers College died April 15, 1950. He was a charter member of the Association.

Mr. G. M. V. Tryon of Fenton, Michigan, died May 1, 1950.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for a three-year term beginning July 1, 1950 by a mail vote of the membership of the Association in the Sections indicated:

Illinois	A. E. GAULT, Bradley University
Iowa	E. N. OBERG, University of Iowa
Louisiana-Mississippi	T. A. BICKERSTAFF, University of Mississippi
Maryland-District of Columbia-Virginia	G. R. CLEMENTS, U. S. Naval Academy
Michigan	J. S. FRAME, Michigan State College
Minnesota	R. H. CAMERON, University of Minnesota
Philadelphia	G. E. RAYNOR, Lehigh University
Southern California	C. G. JAEGER, Pomona College
Texas	E. H. HANSON, North Texas State College

Great interest is shown by our members in these elections for Sectional Governors. In eight of the nine Sections more than half of those eligible to vote cast their votes in time to be counted. The highest percentage of votes cast was 70 per cent in the Louisiana-Mississippi Section.

It should also be noted that each vote is important in these elections. In the case of two Sections, the elected Governor received one more vote than the losing candidate. In the case of a third Section where there were more than two nominees, the winning candidate received two more votes than two other candidates who were tied for second place.

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 119 persons have been elected to membership by the Board of Governors on applications duly certified:

L. J. ABBEDUTO, Student, Illinois Institute of Technology, Chicago, Ill.

WALTER ABRAMOWITZ, Student, Polytechnic Institute of Brooklyn, N. Y.

R. L. ADEY, Student, Oregon State College, Corvallis, Ore.

KANHAYALAL AGGARWAL, B.A. (Panjab) Proprietor, Messrs. Hariram Aggarwal & Sons, Majith Mandi, Amritsar, India

A. N. AHEART, A.M. (Harvard) Instructor, West Virginia State College, Institute, W. Va.

L. W. AKERS, M.A. (Kansas) Asso. Professor, Central Missouri College, Warrensburg, Mo.

W. H. BARTELS, B.S. in E.E. (Michigan S.) Assistant Electrical Engineer, Board of Water and Light Commission, Lansing, Mich.

ALEXANDER BASIL, Student, Hofstra College, Hempstead, N. Y.

D. C. BENSON, Student, Pomona College, Claremont, Calif.

JONAS BERARU, B.A. (U.C.L.A.) Research Assistant, North American Aviation, Inc., Downey, Calif.

GERALD BERMAN, M.A. (Toronto) Teaching Fellow, University of Toronto, Ont.

A. F. BERNDT, Student, Cooper Union, New York, N. Y.

C. C. BRAUNSCHWEIGER, Student, Alfred University, N. Y.

H. O. BREMER, B.A. (Hofstra) 46—21 156 Street, Flushing, N. Y.

E. B. BRIDGFORTH, A.B. (Duke) Grad. Student, University of Chicago, Ill.

M. D. R. BROWN, Student, University of Kansas, Lawrence, Kans.

MARGUERITE A. BROWN, Student, Carleton College, Northfield, Minn.

C. N. CAMPOPIANO, B.S. (Rutgers) Grad. Student, Rutgers University, New Brunswick, N. J.

DOROTHY I. CARPENTER, M.A. (Michigan) Instructor, Denison University, Granville, Ohio.

R. E. CARROLL, M.S. (Illinois) Instructor, Emory University, Ga.

B. R. CATO, JR., A.M. (Duke) 2032 Greenway Avenue, Charlotte, N. C.

I. J. CHERRY, B.S. (Oregon S.C.) Grad. Student, Oregon State College, Corvallis, Ore.

H. J. CHESTON, JR., M.A. (Maryland) Instructor, Montgomery Junior College, Bethesda, Md.

P. F. CLEMENS, M.A. (New York S.C.T., Albany) Instructor, Rensselaer Polytechnic Institute, Troy, N. Y.

M. J. CLEVELAND, M.S. (Florida) Instructor, University of Florida, Gainesville, Fla.

H. M. COCHRAN, Col., U. S. A., 101 Miramonte Avenue, Palo Alto, Calif.

D. A. DODGE, Student, University of Kansas, Lawrence, Kans.

R. F. DURIG, Student, Kent State University, Ohio.

JAMES EASTCOTT, Student, McMaster University, Hamilton, Ont.

C. C. FAITH, Student, University of Kentucky, Lexington, Ky.

H. M. FELDMAN, Ph.D. (Washington U.) Lecturer, Washington University; Teacher, Board of Education, St. Louis, Mo.

A. M. FEYERHERM, M.S. (Iowa) Instructor, Iowa State College, Ames, Iowa.

F. H. FISHER, A.B. (Washington & Jefferson) Teaching Fellow, West Virginia University, Morgantown, W. Va.

W. L. FRANK, Student, Illinois Institute of Technology, Chicago, Ill.

R. F. GABRIEL, M.A. (Columbia) Instructor, St. Francis College, Brooklyn, N. Y.

W. V. GAMZON, M.A. (U.C.L.A.) Teacher, Washington Jr. High School, Long Beach, Calif.

ROBERT GARFUNKEL, Student, Rutgers University, New Brunswick, N. J.

ISIDORE GOLDMAN, B. of M.E. (Cooper Union) Mechanical Engineer, New York Naval Shipyard, Brooklyn, N. Y.

W. T. GRAYBEAL, A.M. (North Carolina) Asst. Professor, Emory and Henry College, Emory, Va.

J. E. HAFSTROM, M.A. (Minnesota) Instructor, University of Minnesota, Duluth, Minn.

W. C. HAMILTON, Student, Oklahoma A. and M. College, Stillwater, Okla.

J. B. HANSEN, Student, Harvard University, Cambridge, Mass.

L. H. HARRIS, Student, Polytechnic Institute of Brooklyn, N. Y.

L. L. HELMS, Student, Bradley University, Peoria, Ill.

N. K. HERSH, M.A. (Kansas) Instructor, Tulane University, New Orleans, La.

R. J. HEYMAN, Student, Illinois Institute of Technology, Chicago, Ill.

P. A. HINRICHS, Student, Hofstra College, Hempstead, N. Y.

R. V. HOGG, M.S. (U. of Iowa) Instructor, State University of Iowa, Iowa City, Iowa.

N. C. HOOVER, M.A. (Kansas) Instructor, Washburn Municipal University, Topeka, Kans.

DAVID HORWITZ, Student, Illinois Institute of Technology, Chicago, Ill.

L. N. HOWARD, Student, Swarthmore College, Pa.

EDWARD KEEGAN, B.S. (Kansas S.T.C.) Grad. Student, Kansas State Teachers College, Pittsburg, Kans.

R. A. KENNEDY, Student, Illinois Institute of Technology, Chicago, Ill.

DON KIRKHAM, Ph.D. (Columbia) Professor, Iowa State College, Ames, Iowa.

J. D. E. KONHAUSER, B.S. (Penna. State) Instructor, Pennsylvania State College, State College, Pa.

JACOB KOREVAAR, Ph.D. (Leiden) Visiting Lecturer, Purdue University, Lafayette, Ind.

S. B. KRAMER, Student, Polytechnic Institute of Brooklyn, N. Y.

H. C. KRANZER, Student, New York University, N. Y.

K. E. LAKE, Student, Washburn University, Topeka, Kans.

A. S. LEE, JR., Student, Southern Methodist University, Dallas, Texas.

FERNAND LEMAY, Student, Laval University, Quebec, P. Q.

KENNETH LEWIS, B.P.(Toronto) Asst. Professor, University of Florida, Gainesvile, Fla.

J. W. LINDSAY, Instructor, Texas Technological College, Lubbock, Texas.

DOROTHY K. LOONEY, Student, Regis College, Weston, Mass.

I. J. LOWE, Student, Cooper Union, New York, N. Y.

A. W. MALL, A.B.(Clark) Instructor, St. Lawrence University, Canton, N. Y.

A. P. MATTUCK, Student, Swarthmore College, Pa.

C. H. MCCALL, JR., Student, George Washington University, Washington, D. C.

JESSIE MCLEAN, M.S.(Washington U.) Teacher, Harris Teachers College, St. Louis, Mo.

K. W. MCVOY, JR., Student, Carleton College, Northfield, Minn.

BERT MENDELSON, B.A.(Columbia) Grad. Assistant, University of Nebraska, Lincoln, Nebr.

R. J. MIHALEK, Student, Illinois Institute of Technology, Chicago, Ill.

BENJAMIN MITTMAN, Student, Illinois Institute of Technology, Chicago, Ill.

J. A. MORALES, M.A.(Florida) Instructor, University of Florida, Gainesville, Fla.

J. H. MULLIGAN, JR., Ph.D.(Columbia) Asst. Professor, New York University, N. Y.

T. A. NEWTON, B.S.(Colorado A. & M.) Grad. Student, University of Georgia, Athens, Ga.

H. A. OSBORN, A.B.(Princeton) Teaching Assistant, Stanford University, Calif.

A. J. OWENS, M.S.(Iowa) Grad. Student, University of Florida, Gainesville, Fla.

W. D. PAXTON, JR., Student, Pomona College, Claremont, Calif.

A. F. PAYTON, Student, Rutgers University, New Brunswick, N. J.

J. M. PIKE, B.S.(Kansas S.T.C.) Grad. Student, Kansas State Teachers College, Pittsburg, Kans.

W. P. REID, Ph.D.(Pittsburgh) Asst. Professor, U. S. Air Force Institute of Technology, Wright Field, Ohio.

G. A. REILLY, B.S.(St. Mary's) Clerk, American Hospital and Life Insurance Co., San Antonio, Texas.

T. L. REYNOLDS, M.A.(North Carolina) Grad. Student and Part-time Instructor, University of North Carolina, Chapel Hill, N. C.

AUDREY I. RICHARDS, M.A.(Columbia) Instructor, Utica College, N. Y.

W. J. RISSLER, A.M.(New Jersey S.T.C.) Instructor, State Teachers College, Frostburg, Md.

VIRGINIA E. ROBERTS, M.A.(Texas Tech.) Instructor, Texas Technological College, Lubbock, Texas.

J. M. ROBERTSON, B.S.(Clemson) Part-time Instructor, University of Florida, Gainesville, Fla.

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JANUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The twelfth annual meeting of the Northern California Section of the Mathematical Association of America was held at the University of California, Berkeley, on Saturday, January 28, 1950. Professor H. M. Bacon, Chairman of the Section, presided.

The attendance was seventy-six including the following forty-seven members of the Association: H. A. Arnold, H. M. Bacon, G. A. Baker, T. J. Bass, Jr., Alice K. Bell, B. A. Bernstein, M. T. Bird, Elizabeth W. Brownell, A. C. Burdette, K. L. Cooke, M. A. Dernham, Brother Dominic, Roy Dubisch, Hazel E. Eggett, E. B. Eilertsen, G. C. Evans, E. A. Fay, S. A. Francis, C. M. Fulton, W. H. Gregory, W. R. Hanson, C. A. Hayes, Jr., Marjorie Ley Hoffman, Mary Thayer Huggins, D. W. Hullinghorst, Free Jamison, J. L. Kelley, Helena G. Kusick, D. H. Lehmer, Sophia McDonald, E. D. Miller, F. R. Morris, W. H. Myers, Andrewa Noble, C. D. Olds, George Polya, Edris Rahn, E. S. Robbins, E. B. Roessler, Kathyrn B. Rolfe, J. B. Rosser, Falka G. Sturges, Mary V. Sunseri, Irving Sussman, Edwin Tabor, L. A. Walker, A. R. Williams.

At the business meeting the following officers were elected for the coming year: Chairman, S. A. Francis, San Mateo Junior College; Vice-Chairman, D. H. Lehmer, University of California, Berkeley; Secretary-Treasurer, E. B. Roessler, University of California, Davis; Representative on the *California Journal of Secondary Education*, Ruth G. Sumner, Oakland High School.

By invitation of the Section, Professor A. W. Tucker of Princeton University gave an address during the morning session.

The program consisted of the following papers:

1. *On the birectangular quadrilateral*, by Professor C. M. Fulton, University

of California, Davis.

A complete set of formulas for a birectangular quadrilateral in hyperbolic plane geometry is obtained by a simple method. Using Cartesian coordinates, these formulas quite naturally lead to the familiar expressions for arc length and area. They also give the "slope," $-\mathrm{ctn}\alpha = \mathrm{sech}\, yD_x y$.

2. *The M.A. degree in mathematics as offered at San Jose State College*, by Professor W. H. Myers, San Jose State College.

A description of the content of this paper has been printed in this MONTHLY in the report of the March 1950 meeting of the Southern California Section of the Association.

3. *Let us teach guessing*, by Professor George Pólya, Stanford University.

It has been said very often, and certainly with good reasons, that teaching mathematics should be used as an opportunity to teach demonstrative reasoning. In the speaker's opinion, teaching mathematics should also be used as an opportunity to teach plausible reasoning. Inductive reasoning is a very important kind of plausible reasoning; (non-mathematical) induction means (roughly) guessing a general law from particular instances, and testing it by examining further particular instances. Many important mathematical results have been discovered by induction, especially in the theory of numbers. In his classes, the speaker tries, with appropriate suggestive questions, to make the students discover inductively the law by themselves. The speaker gave a practical demonstration of how this is done.

4. *Games of strategy*, by Professor A. W. Tucker, Princeton University.

Two persons play the following game a large number of times. Each chooses independently one of the ten numbers $1, 2, \cdots, 10$, and the two chosen numbers are then compared: the chooser of the higher number pays the chooser of the lower number one penny or vice versa, according as the difference between the numbers is odd or even—there being no payment if the numbers are the same. *Problem:* What long-run "strategy" should each player formulate to protect himself on the average against loss? *Solution:* Each should choose the nine numbers $1, 2, \cdots, 9$ at random with equal frequencies, and never choose 10. If the game is varied by making the payoff two pennies when the difference is even, it turns out that the optimal strategy consists in choosing 8, 9, 10 (only) at random with relative frequencies 1:2:1. These are simple examples of *two-person zero-sum games* (see von Neumann and Morgenstern, *The Theory of Games and Economic Behavior*, Princeton, 1943). The main theorem for such games is that there always exist optimal strategies for the two players, a strategy for the one player that maximizes his minimum expectation, and a strategy for the second player that minimizes his maximum risk, the max-min equaling the min-max. A proof of this fundamental theorem has recently been found within a direct extension of the algebra of simultaneous linear equations to simultaneous linear inequalities.

Discussions for the general reader of economic and military implications of von Neumann's theory of games have been given by Oskar Morgenstern in the *Scientific American*, May 1949, pp. 22–25, and by John McDonald in *Fortune*, June 1949, pp. 100–110. Other references are: R. A. Fisher, *Mathematical Gazette*, 18 (1934), pp. 294–297; R. Bellman and D. Blackwell, *Proceedings of the National Academy of Science*, 35 (1949), pp. 600–605; and various authors, *Annals of Mathematics Study* No. 24 (1950).

This paper arises from work conducted under Office of Naval Research Contracts at Princeton and Stanford Universities.

5. *Note on elementary function theory*, by Professor J. L. Kelley, University of California, Berkeley.

A proof of a form of the Cauchy theorem is given which, though precise, avoids topological assumptions.

The basic lemma: If f has a continuous complex derivative in an open set G, and P, a func-

tion on the unit square with values in G, has piecewise continuous second partial derivatives, then:

$$\frac{d}{ds}\int_0^1 f[P(s,t)]P_t(s,t)dt = \int_0^1 \{f'[P(s,t)]P_t(s,t)P_s(s,t) + f[P(s,t)]P_{ts}(s,t)\}dt$$

$$= f[P(s,t)]P_s(s,t)\Big|_{t=0}^{t=1}.$$

This lemma implies directly that deformation of a path p in a domain where f is analytic does not change the value of $\int_p f(z)dz$ if either: (a) the end points of p are fixed under the deformation; or (b) the deformation is by means of closed curves.

These two rules, together with the following obvious statement, are sufficient for all ordinary calculations with contour integrals. If p and q are paths, and the end point of p is the beginning point of q, and if $\bar{p}\bar{q}$ is the path obtained by traversing first p and then q, then

$$\int_{\bar{p}\bar{q}} f(z)dz = \int_p f(z)dz + \int_q f(z)dz.$$

6. *A classroom note*, by Professor C. D. Olds, San Jose State College.

The author discusses an interesting connection between E. T. Whittaker's method for finding the roots of power series (see, for example, this MONTHLY, vol. 49, 1942, p. 462), and some of the recent modifications of Newton's method (H. S. Wall, this MONTHLY, vol. 55, 1948, p. 90). In particular, the approximate formula given by Wall for finding the n-th root of a positive number is obtained directly from the first two terms of Whittaker's formula.

7. *A slant on the problem of crossed ladders*, by Professor H. A. Arnold, University of California, Davis.

Two ladders of lengths a and b connect the bases of parallel vertical walls, z units apart, with points at heights x and y on the opposite walls. They cross at height unity and at distance l from the "y wall." It is required to find z. It is shown that

$$\frac{l}{z} = \frac{1}{x} = \frac{y-1}{y}; \qquad y^2 - x^2 = b^2 - a^2 = K; \qquad x + y = xy.$$

Set $x+y=4u$, $y-x=v$. Then $u=1+(K^2/256)\,(1/u^3)$, a form suitable for computation by successive substitutions. Finally, $v=y-x=(y^2-x^2)/(y+x)=K/4u$, and x and y may be computed from u and v.

E. B. ROESSLER, *Secretary*

MARCH MEETING OF THE MISSOURI SECTION

The Missouri Section of the Mathematical Association of America met at Washington University in St. Louis on Friday afternoon, March 24, 1950, and Saturday morning, March 25, 1950. Professor G. W. Ewing presided at the Friday session, and Professor C. W. Mathews, Chairman of the Section, presided at the Saturday session.

Sixty persons were in attendance including the following thirty-four members of the association: F. L. Barrow, S. Louise Beasley, L. M. Blumenthal, E. W. Bold, S. S. Cairns, J. L. Collins, Mary L. Cummings, R. E. Ekstrom, G. M. Ewing, E. A. Goodhue, Franklin Haimo, F. F. Helton, I. I. Hirschman, Jr., C. A. Johnson, L. O. Jones, Lois Karr, R. E. Lee, Walter Leighton, Ella Martha, C. W. Mathews, Jr., R. R. Middlemiss, Marie A. Moore, R. M. Rankin, Francis Regan, P. R. Rider, Sister Mary Teresine, R. L. Snider, W. L. Stanley, W. J.

Thron, Choy-Tak Taam, W. R. Utz, Jr., W. A. Vezeau, Margaret F. Willerding, W. D. Williams.

The following officers were elected for the coming year: Chairman, F. F. Helton, Central College; Vice-Chairman, L. O. Jones, William-Jewell College; Secretary-Treasurer, Margaret F. Willerding, Harris Teachers College. It was announced that the 1951 meeting will be held at Central College in the spring of 1951. Following the Friday afternoon meeting, tea and coffee were served in the Lounge of the Women's Building.

The following papers were presented:

1. *The geometry of polyhedra*, by Professor S. S. Cairns, University of Illinois.

Professor Cairns was a guest speaker. His paper was expository, and contained a few simple and, for the most part, old results concerning the structure of polyhedra. Despite their simplicity, these results are not widely known, partly because they are out of the main current of traditional geometry, and partly because they are special cases of more general theorems, proved by relatively sophisticated methods. Some of the interest in them stems from the fact that they are denials of plausible statements. Thus, for example, no polyhedron exists with exactly seven edges, though all other numbers greater than five are possible. No polyhedron exists all of whose faces have more than five edges. If none of the faces of a polyhedron are triangular, then some vertex belongs to exactly three faces. No polyhedron has an odd number of odd-sided faces. These and related results were established by elementary arguments, well within the grasp of a high school student.

2. *Notes on Martin's ergodic function*, by Professor W. R. Utz, University of Missouri.

This paper will be published in a subsequent issue of this MONTHLY.

3. *Symmetry in four dimensions*, by Mr. E. W. Bold, St. Louis University

Basic ideas involving the geometry of four dimensions are explained, after which definitions concerning symmetry with respect to a point, line, plane, and hyperplane are considered. Theorems on symmetry in four dimensions are developed by methods of analysis situs. In particular, theorems on perpendicular and coincident elements, as well as those on parallel elements, are discussed.

4. *Hurwitz polynomials in engineering mathematics* by Professor Herman Betz, University of Missouri, introduced by the Secretary.

The purpose of this paper is to call attention to the great importance of Hurwitz polynomials in questions of stability arising from engineering problems. It is shown how the well known Hurwitz criteria or their equivalent may be obtained for those cases which engineers are mostly concerned with, using however, quite elementary methods which are well within the grasp of engineers.

5. *Distribution law of the product of two variables independently distributed in Pearson's type I laws*, by Mr. John S. Hagen, St. Louis University, introduced by the Secretary.

Two variables x and y are known to be independently distributed according to Pearson's Type I laws. In the joint frequency function of x and y, the substitution $y=Z/x$ is made, and the variable x is integrated out, giving a function of the product xy. The conditions under which the series in $f(Z)$ is finite, the general appearance of the curve, and some of the properties of the functions, are discussed.

6. *On quasi-analytic functions of analytic functions*, by I. I. Hirschman, Jr., Washington University.

In the theory of quasi-analytic functions it is sometimes necessary to obtain bounds on the derivatives of $f[g(x)]$ from bounds on the derivatives of f and g separately. Several theorems of this type are presented. It is shown in particular that if g is analytic, and if we possess information concerning the behavior of $g(x)$ in the complex plane, then very precise estimates may be found.

7. *A method of uniformizing grades*, by Marlow Sholander, Washington University, introduced by the Secretary.

The speaker presented a description of the method which has been used with some success at Washington University in multiple section mathematics courses, to reduce the probability that two students with different instructors receive grades incommensurate with their ability and effort.

MARGARET F. WILLERDING, *Secretary*

APRIL MEETING OF THE METROPOLITAN NEW YORK SECTION

The April meeting of the Metropolitan New York Section of the Mathematical Association of America was held at the City College of the College of the City of New York, New York, on Saturday, April 1, 1950. The Chairman of the Section, Professor B. P. Gill, introduced President Wright, and presided at the business meeting which preceded the afternoon session; the High School Vice-Chairman, Mr. Alan Wayne, presided at the morning session; the Collegiate Vice-Chairman, Professor L. F. Ollmann, presided at the afternoon session.

One hundred and ten persons attended the meeting, including the following sixty-seven members of the Association: M. W. Al-Dhahir, Brother Bernard Alfred, R. C. Archibald, F. C. Beckman, Samuel Borofsky, C. B. Boyer, Paul Brock, A. B. Brown, J. H. Bushey, J. J. Clark, T. F. Cope, W. H. H. Cowles, P. M. Curran, I. A. Dodes, J. N. Eastham, W. H. Fagerstrom, H. F. Fehr, B. P. Gill, G. C. Helme, A. J. Hoffman, E. Marie Hove, T. R. Humphreys, Solomon Hurwitz, L. C. Hutchinson, Alto C. Juelich, C. J. Kaufman, L. S. Kennison, G. A. Keyes, H. S. Kieval, R. J. Kohlmeyer, H. C. Kranzer, C. H. Lehmann, D. R. Lintvedt, E. R. Lorch, D. M. MacEwen, V. S. Mallory, D. May Hickey Maria, F. H. Miller, A. J. Mortola, D. S. Nathan, M. A. Nordgaard, Eugene Odin, C. S. Ogilvy, L. F. Ollmann, O. L. Phillips, W. L. Pickard, E. L. Post, Walter Prenowitz, J. J. Quinn, Moses Richardson, Selby Robinson, H. D. Ruderman, John Salerno, Abraham Schwartz, Aaron Shapiro, James Singer, M. G. Smith, C. G. Solky, E. P. Starke, Mildred M. Sullivan, Annita Tuller, H. E. Wahlert, Alan Wayne, W. W. Winnis, J. M. Wolfe, H. N. Wright, H. J. Zimmerberg.

The following officers were elected for the coming year: Chairman, Brother Bernard Alfred, Manhattan College; Collegiate Vice-Chairman, James Singer, Brooklyn College; High School Vice-Chairman, I. A. Dodes, Stuyvesant High School; Secretary, H. S. Kieval, Brooklyn College; Treasurer, Aaron Shapiro, Midwood High School. Professor W. H. Fagerstrom, Chairman of the Committee on Prizes and Awards, reported on the progress of the Committee and on the forthcoming High School Contest sponsored by the Section. Complete de-

tails of the contest will appear in this MONTHLY.

Dr. H. N. Wright, President of The City College, welcomed the people at the meeting, and then the following papers were presented:

1. *What is mathematical functionality?* by Dr. Barnett Rich, Richmond Hill High School (introduced by Mr. Alan Wayne).

In this paper, the author condensed material contained in his dissertation, which pertained especially to the problems of what constitutes the domain of the subject matter of functionality, and how the basic concepts of functionality or relational variability may be applied to the teaching of mathematics, and utilized further in the teaching of science. The main points treated in the paper concern: (1) The use of a new symbol, namely $\mathfrak{x}$, for the ratio of any two ordered values of a variable; (2) A comparison of the similarity of the uses of the new symbol $\mathfrak{x}$ to those of Δx; (3) The approach to functionality through the use of three-variable relationships of the simplest operational character, such as $z=xy$, $z=x+y$, and $z+x^y$; (4) The emphasizing of sense change and variation in the teaching of both mathematics and science; (5) Continuity in the instruction of functionality from the elementary school to the college.

2. *Inequalities and convex bodies*, by Professor E. R. Lorch, Columbia University.

Basic to the development of analysis are certain inequalities such as those of Cauchy-Schwartz, Holder, and the triangle inequality. The relation between inequalities and the theory of convex bodies has been known since the time of Minkowski. However, the classic theory merely has existential force, and does not give methods for actual computation. This paper presents precise and analytically transparent methods for the derivation of all inequalities of this type. The program is based on a new definition of convexity in which the notions of differentiability, homogeneity, definite quadratic form, and one-to-one transformation play a dominant role. It is shown that bodies which are convex in the classic sense may be approximated arbitrarily closely by those of our type. This circumstance makes it possible to reconsider every problem on convexity along lines in which these new techniques apply.

3. *What mathematics should a high school teacher know?* by Professor H. F. Fehr, Teachers College, Columbia University.

Teacher examinations in large cities demand a knowledge of advanced mathematics of little use in high school teaching. State certifications require from zero to twenty semester hours of mathematics; college major requirements vary from eighteen to forty-six semester hours. Commission reports recommend study beyond the calculus and in the physical sciences. The changing purpose of high school instruction demands other mathematical knowledge than traditional preparatory mathematics. Beyond high school subjects the teacher must know analytic geometry, advanced algebra, the calculus, foundations of modern mathematics, and applications and the history of what he teaches. There is special need for professionalization of high school mathematics, that is, a study from an advanced view-point, and from many avenues of approach, of the high school content. The high school teacher must be a scholar within his field, not necessarily beyond it.

4. *What makes a good problem?* by Professor E. P. Starke, Rutgers University.

Several possible characteristics of a "good" problem are proposed, discussed briefly, and illustrated by selections from the problem departments of this MONTHLY and other journals. A good problem meets one or more of the following requirements: it may extend old results or open up a new field of interest; it may be nicely adapted to develop facility and ingenuity for its solution; it may force out into the open certain misunderstandings or inadequacies; it may have an

unexpected and surprising conclusion; it may require the invention of new methods of attack; or it may possess a wide range of appeal, perhaps through being dressed up in picturesque language. A real need exists for more good problems both for class practice and for the pages of the problem departments.

5. *Analog computers*, by Mr. A. Goetz, Arma Corporation (introduced by Mr. Eugene Odin).

The speaker stressed the role of the machine computer with regard to its utility in the solution of present day problems. He classified the computers into several types, and chose the automatic electromagnetic analog computer for closer examination. The components of such a device were enumerated. A review of the operations of arithmetic, calculus, and function generation was made, and illustrations of various instrumentations of the operations of addition, subtraction, multiplication, division, differentiation, integration, and function generation were given.

The speaker then developed the block diagram of a computer to solve the simple geometric problem of a plane triangle where two sides and the included angle are known and the length of the other side is required.

The hope was expressed that the dissemination of information regarding the capabilities of the analog computer would result in its application in fields as yet unexplored.

JAMES SINGER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-fourth Annual Meeting, University of Florida, Gainesville, December 30, 1950.

Thirty-second Summer Meeting, University of Minnesota, Minneapolis, September 3–4, 1951.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, May, 1951.

ILLINOIS, University of Illinois, Urbana, May 11–12, 1951.

INDIANA, May 5, 1951.

IOWA, Wartburg College, Waverly, April 20–21, 1951.

KANSAS

KENTUCKY, Eastern Kentucky State College, Richmond, April 28, 1951.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, December 9, 1950.

METROPOLITAN NEW YORK, Spring, 1951.

MICHIGAN, East Lansing, March 24, 1951.

MINNESOTA, Duluth Branch of University of Minnesota, October 7, 1950.

MISSOURI, Central College, Fayette, Spring, 1951.

NEBRASKA, University of Nebraska, Lincoln, May 5, 1951.

NORTHERN CALIFORNIA, University of San Francisco, January 27, 1951.

OHIO, April 21, 1951.

OKLAHOMA, Oklahoma City, November 13, 1950.

PACIFIC NORTHWEST, State College of Washington, Pullman, June 15, 1951.

PHILADELPHIA, Lehigh University, Bethlehem, Pennsylvania, November 25, 1950.

ROCKY MOUNTAIN, Colorado State College of Education, Greeley, April, 1951.

SOUTHEASTERN, Vanderbilt University and Peabody College, Nashville, Tennessee, March 16–17, 1951.

SOUTHERN CALIFORNIA, Whittier College, Whittier, March 10, 1951.

SOUTHWESTERN, University of New Mexico, Albuquerque, Spring, 1951.

TEXAS, Southern Methodist University, Dallas, Spring, 1951.

UPPER NEW YORK STATE, Hamilton College, Clinton, Spring, 1951.

WISCONSIN, Carroll College, Waukesha, May, 1951.

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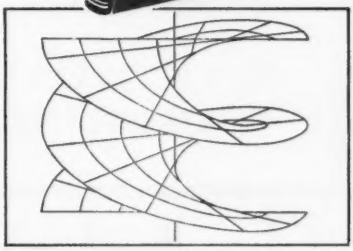

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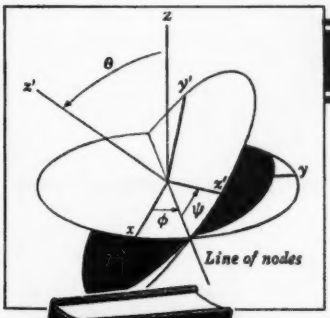

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